A VORTEX MODEL DEALING WITH THE AIRSTREAM AT THE ROTOR BLADE OF A HELICOPTER

W. H. Isay

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W. H. Isay

ABSTRACT: With the aid of the non-stationary vortex line--support theory an approximate method is given for calculating the buoyancy distribution at rotor blades in forward flight. The presented theory takes into consideration, with reference to vortex geoemetry, a non-uniform (especially trapezoidal) flow through the plane of the rotor, as well as the unrolling process and the contraction of the free transverse vortex. In the determination of the circulation distribution of the rotor airflow, a special case is handled where the blades penetrate the unrolled and contracted tip vortex and root vortex of the sometimes forward-driving airfoils. The appendix contains formulas for computing acoustic pressure fields radiated from a rotor (compressible treatment).

1. Introduction And Survey

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Various approximate methods are given in the literature [1, n.b. Ch. III A] for calculating the airstream at the rotor blades which are applicable to the management of various flying conditions (for example, periodic or aperiodic), depending on the extent to which these are neglected. Nevertheless, none of the results put forward by any of these methods are completely satisfactory, so that it appears desirable to become concerned with further improvments, especially the consideration of the neglected effects.

From a theoretical point of view, the linearized expanded line-support theory is propounded most consistently when its included simplifications (among others, neglect of the deformation of free vortex surfaces owing to the unequal rotor flow-through and the strike movement of the blades, the contraction and unrolling process of the free transverse vortex, the inclined and backward airflow of the blades at the larger degrees of advance) also make the most of considerable separation from the physical reality [1, 3].

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*Numbers in the margin indicate pagination in the foreign text.

In the work submitted here a modified vortex model is presented based on the line-support theory. This allows some of the omitted effects in the analyses to be understood. The unequal flow through the plane of the rotor at one time was part of this purpose. Emphasis should be placed, however, on the influence the unrolled and contracted tip vortex and root vortex of the advance-traveling blades have on the buoyancy distribution of the wing when the observed airfoil space point strikes across them.

The airflow boundary condition at the rotor blade thereby leads to integral equations with novel singularities, whose resolution theory constitutes an essential part of the work presented here. /2

2. Concept of the Vortex Model

The rotor under consideration has N airfoils (n = 0, 1,...N - 1) and exists in an absolutely periodic flying condition with a velocity w_o in the direction of the negative z-axis. Let the n = 0 blade be the airfoil space point on which we will satisfy the airflow boundary condition. In our aircraft-fixed coordinate system, let the plane of the rotor be x = 0[1], w_o therefore is the airflow velocity in the z-direction, and furthermore, with $u_o^2 << w_o^2$, u_o is an airflow in the x--direction (Figure 1).

The free transverse vortex and longitudinal vortex turn out to be, as in the linearized theory [1, 2, 3], within an infinitely inclined extended cylinder behind the rotor.

Due to the geometry of the free vortex surface, different hypotheses, suggested by physical observation, are put forward for the individual rotor airfoil and flowing-off zone.

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[1]As coordinates we use either cartesian x, y, z or cylindrical coordinates x, y = r cos ϕ, z = r sin ϕ. In a similar manner, as integration variables for vortex lines, we use s instead of r and ψ instead of ϕ. ω is the angular velocity and $\phi_n = \phi_o + 2\pi n/N$ characterizes the momentary angle position of the n-th blade. R_i and R_o are inner and outer radii of the blades. Finally, we designate by $\omega = u\omega_x + v\omega_y + w\omega_t = u\omega_x + V\omega_\varphi + W\omega_r$ as usual, the absolute velocity based upon the aircraft-fixed coordinate system.

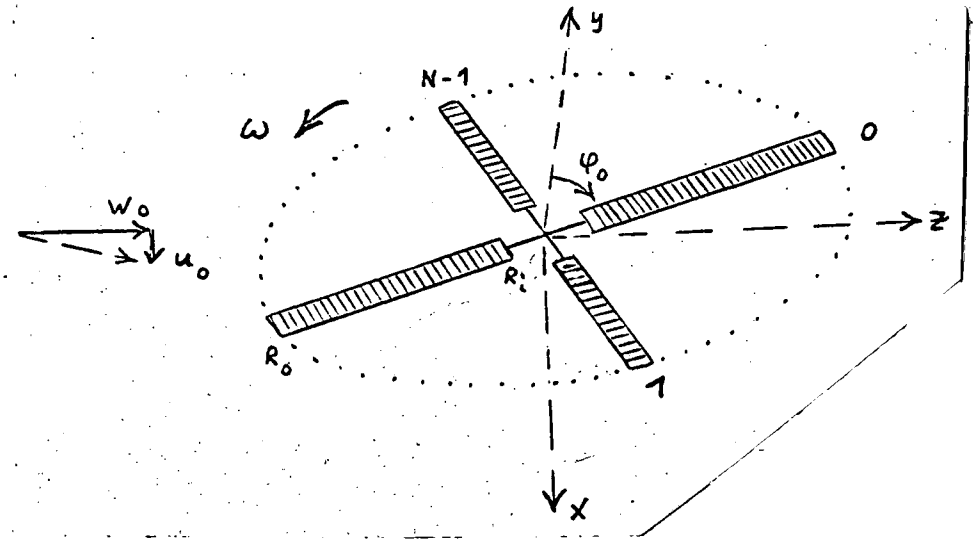

Figure 1. Coordinate System of the Rotors (N = 4).

Without the contraction and the unrolling process to bear in mind, trapezoidal flow-through sets up through the plane of the rotor in the x-direction, perhaps by:

$$k_0(s,\psi) = k_{00} + k_{01}\frac{S}{R_0}\sin\psi,\quad k_{00} = \text{const};\quad k_{01} = \text{const}; \tag{1}$$

on free vortex surfaces, in the form of (Figure 2) [1, 2]:

$$\omega_f = \psi\, k_0(S_1\varphi_n + \psi)\,\omega_x + s\cos(\varphi_n + \psi).\ \omega_y + [S\sin(\varphi_n + \psi) + k_*\psi]\omega_t.$$

with

$$k_* = \text{const};\quad (0 \leqslant \psi \leqslant \infty;\ R_i \leqslant s \leqslant R_0). \tag{2}$$

$$k_* \approx s\,\frac{W_0 + W_Q + W_L}{W_S + V_Q + V_L} \approx \frac{W_0}{W}.$$

At constant flow-through, k_o = constant is assumed. If we find, generally, /3 no vortex motion through the plane of the rotor in certain zones, then we have to assume instead that $k_o = 0$ in that case.

The vortex axis vector of the free transverse vortex is given by $d\sigma_Q = \frac{\partial H_f}{\partial\psi}d\psi$, that of the free longitudinal vortex by $d\sigma_L = \frac{\partial H_f}{\partial s}ds$.

We calculated the induced velocity field from the vortex system of the rotor along a line at 3/4 sideview depth of the airfoil space point as it

becomes necessary to the performance of the airflow boundary condition, according to the extended line-support theory (Figure 3).

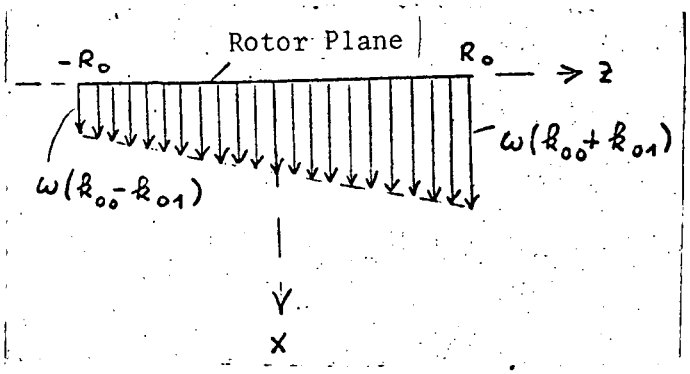

Figure 2. Trapezoidal Flow-Through.

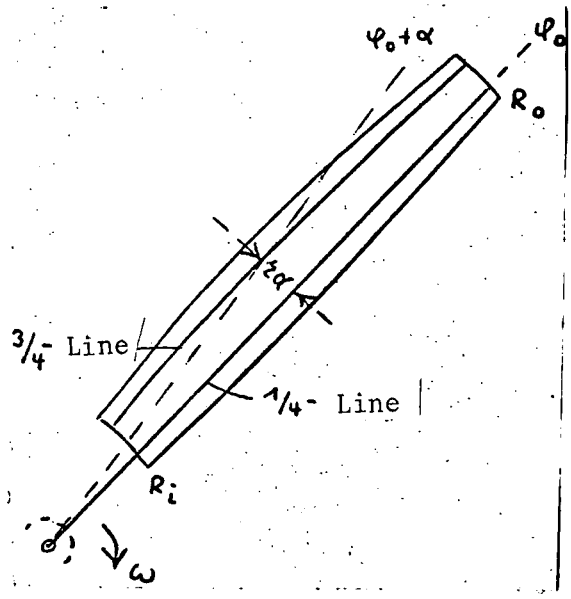

Figure 3. Designations at the Airfoil Blade.

The airflow blades have the depth $2r\alpha(r)$, and its 3/4-line is given by $x = 0$; $y = r\cos(\phi_n + \alpha)$, $Z = r\sin(\phi_n + \alpha)$. $\phi_n = \phi_o + \frac{2\pi n}{N}$. (n = 0 airfoil airfoil space point.

a) We compensate the airfoils in the x = 0 plane of the rotor at the 1/4-line airfoil depth with regulated, bound rod vortices having the circulation $\Gamma(s, \phi_n)$. (n = 0, 1,..., N - 1). (Figure 3). The induced velocity field along the 3/4-line of the airfoil space point is given by:

$$w_\Gamma = \frac{1}{4\pi}\sum_{n=0}^{N-1}\int_{R_i}^{R_o}\Gamma(s,\varphi_n)\frac{r\sin(\alpha-\frac{2\pi n}{N})\cdot w_x\cdot ds}{\sqrt{r^2+s^2-2rs\cos(\alpha-\frac{2\pi n}{N})}^3} \quad (3)$$

b) For the free transverse vortex we select the following model: At the n = 0 blade (airfoil space point) it is assumed that the transverse vortices remain without flow-through at the x = 0 plane of the blade (linearized airfoil theory (up to perhaps 1/4 of the blade depth behind the airfoil (therefore

up to $\psi \approx 2\alpha$), and thereafter leave behind the plane of the rotor[2] (Figure 4) corresponding to the trapezoidal flow-through distribution (1)[2], (Figure 4).

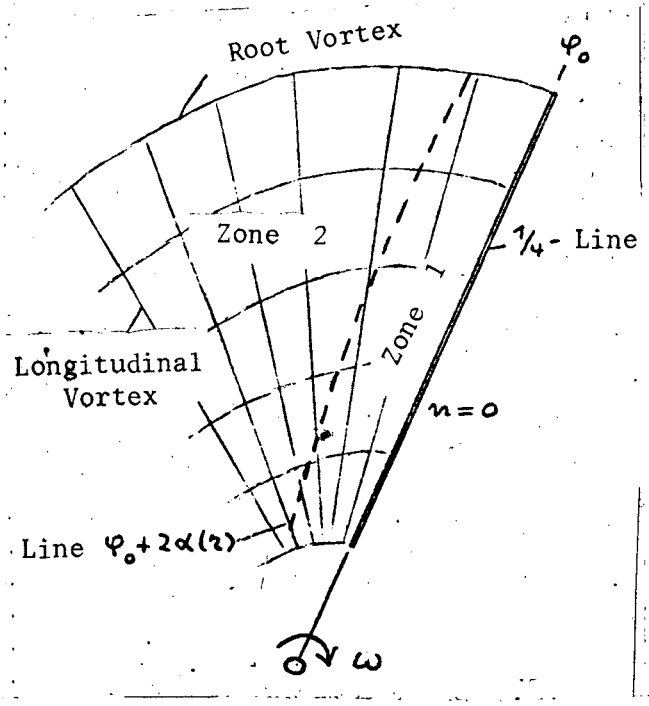

Figure 4. Vortex System at Airfoil Space Point. Zone 1: No flow--through; Zone 2: Trapezoidal flow-through.

At the remaining airfoils ($n \neq 0$) the transverse vortices are retained as though already unrolled in a sharp vortex (intensity + $\Gamma_m(\phi_n + \psi)$, regulated at $S = S_n^{(o)} < R_o$) and in a root vortex (intensity - $\Gamma_m(\phi_n + \psi)$ regulated at $S = S_n^{(i)} > R_i$). Γ_m is thereby the maximum value possessed by the circulation bound to the momentary angular position $\phi_n + \psi$ $S_n^{(o)}$ and $S_n^{(i)}$ of a known wave contraction can be adjusted without difficulty, probably from experiments (Figure 5). / 4

At the n = N - 1 airfoil, which drives the observed airfoil space point forward, we assume that tip vortices and root vortices up to perhaps 1/4 blade depth behind the airfoil space point remain the same in the plane of the rotor (Figure 6), and only then become blown away in the x-direction corresponding to the trapezoidal flow-through (1). This vortex model suggests from practical experience that the tip vortex and root vortex of the forward--driving n = N - 1 blade have a large influence on the air flow rate close to the investigated n = 0 airfoil. That is to say, the latter penetrates frequently throughout the other unrolled free vortex. To investigate the influence of these penetration effects on the buoyancy distribution of a rotor blade is

[2]In place of the maximal value another expected circulation value better suited to the real flow ratios can also be inserted for Γ_m. (For example, as gathered from measurements). In place of a trapezoidal, any other flow-through distribution can also be taken as the basis.

certainly an essential component of the work under discussion. The described vortex geometry mentioned above proves to be especially suited to this. For the tip vortex and root vortex of the remaining n = 1 to n = N - 2 airfoils a flow-through is assumed from the very first, according to equation (1). The influence of this vortex on the buoyancy at the n = 0 blade is smaller, and penetration effects are hardly likely to occur.

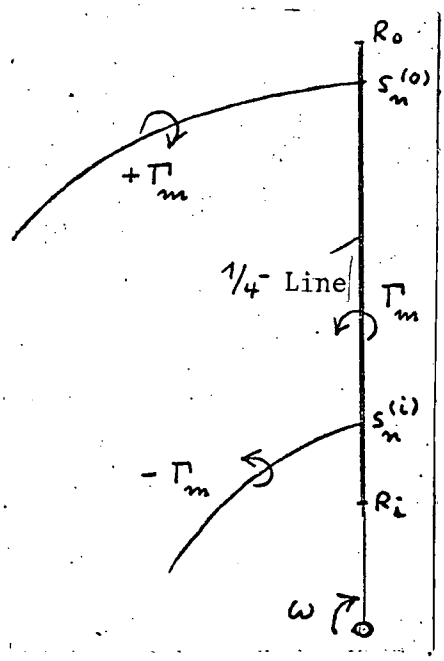

Figure 5. Blade ≠ 0. Unrolled transverse vortex.

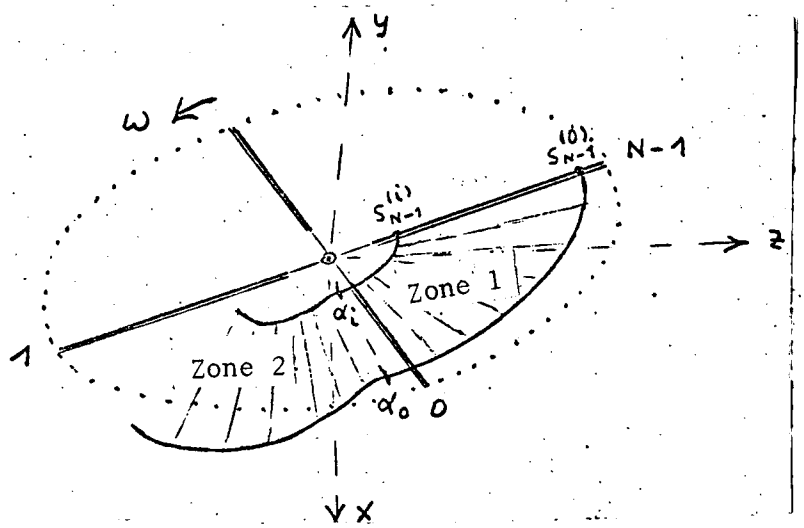

Figure 6. Tip Vortex and Root Vortex of Blade n = N - 1. Zone 1: No flow--through; Zone 2: Trapezoidal flow--through. Analogous longitudinal vortex.

Under these circumstances the induced velocity field from the free transverse vortex on the 3/4-line of the airfoil space point x = 0, y = r cos (ϕ_0 + α), z = r sin (ϕ_0 + α) appears in the following form:

$$w_Q = \sum_{n=0}^{N-1} w_Q^{(n)} \cdot$$

$$w_Q^{(N-1)} = \frac{1}{4\pi} \int_0^{\psi_0^{(0)}+\alpha_0} \Gamma_m(\varphi_0 - \tfrac{2\pi}{N} + \psi) \Big[\big(r\cos(\varphi_0+\alpha) - s_{N-1}^{(0)} \cos(\varphi_0 - \tfrac{2\pi}{N} + \psi)\big)^2 +$$

$$+ \big(r \sin(\varphi_0+\alpha) - s_{N-1}^{(0)} \sin(\varphi_0 - \tfrac{2\pi}{N} + \psi) - k_* \psi\big)^2 \Big]^{-3/2} \cdot \Big[\big(s_{N-1}^{(0)}\big)^2 -$$

$$- 2 S_{N-1}^{(0)} \cos(\alpha - \psi + \tfrac{2\pi}{N}) - k_x 2 \cos(\varphi_0 + \alpha) + k_x S_{N-1}^{(0)} \cos(\varphi_0 - \tfrac{2\pi}{N} + \psi) +$$
$$+ k_x S_{N-1}^{(0)} \psi \sin(\varphi_0 - \tfrac{2\pi}{N} + \psi) \Big] d\psi \cdot n_x -$$
$$- \frac{1}{4\pi} \int_0^{\psi_0^{(i)} + \alpha_i} \Gamma_m(\varphi_0 - \tfrac{2\pi}{N} + \psi) \Big[\text{Integrand with } \Big| S_{N-1}^{(i)} \Big] d\psi \cdot n_x +$$
$$+ \frac{1}{4\pi} \int_{\psi_0^{(0)} + \alpha_0}^{\infty} \Gamma_m(\varphi_0 - \tfrac{2\pi}{N} + \psi) \Big[(\chi_{N-1}^{(0)})^2 + \big(2\cos(\varphi_0 + \alpha) - S_{N-1}^{(0)} \cos(\varphi_0 - \tfrac{2\pi}{N} + \psi) \big)^2 +$$
$$+ \big(2 \sin(\varphi_0 + \alpha) - S_{N-1}^{(0)} \sin(\varphi_0 - \tfrac{2\pi}{N} + \psi) - k_x \psi \big)^2 \Big]^{-3/2} \cdot$$

(4)

$$\cdot \Big\{ \Big[(S_{N-1}^{(0)})^2 - 2 S_{N-1}^{(0)} \cos(\alpha - \psi + \tfrac{2\pi}{N}) - k_x 2 \cos(\varphi_0 + \alpha) + k_x S_{N-1}^{(0)} \cos(\varphi_0 - \tfrac{2\pi}{N} + \psi) +$$
$$+ k_x S_{N-1}^{(0)} \psi \sin(\varphi_0 - \tfrac{2\pi}{N} + \psi) \Big] n_x + \Big[\big(S_{N-1}^{(0)} \sin(\varphi_0 - \tfrac{2\pi}{N} + \psi) - 2 \sin(\varphi_0 + \alpha) +$$
$$+ k_x \psi \big) \frac{\partial}{\partial \psi} \chi_{N-1}^{(0)} - \big(k_x + S_{N-1}^{(0)} \cos(\varphi_0 - \tfrac{2\pi}{N} + \psi) \big) \chi_{N-1}^{(0)} \Big] n_y +$$
$$+ \Big[\big(2 \cos(\varphi_0 + \alpha) - S_{N-1}^{(0)} \cos(\varphi_0 - \tfrac{2\pi}{N} + \psi) \big) \frac{\partial}{\partial \psi} \chi_{N-1}^{(0)} - S_{N-1}^{(0)} \sin(\varphi_0 - \tfrac{2\pi}{N} - \psi) \chi_{N-1}^{(0)} \Big] n_z \Big\} \cdot$$
$$\cdot d\psi - \frac{1}{4\pi} \int_{\psi_0^{(i)} + \alpha_i}^{\infty} \Gamma_m(\varphi_0 - \tfrac{2\pi}{N} + \psi) \Big[\text{Integrand with } \Big| S_{N-1}^{(i)} \Big] d\psi \; ;$$

with $\chi_{N-1}^{(0)} = (\psi - \psi_0^{(0)} - \alpha_0) k_0(S_{N-1}^{(0)}; \varphi_0 - \tfrac{2\pi}{N} + \psi)$; analog $\chi_{N-1}^{(i)}$.

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$$\sum_{n=1}^{N-2} \mathcal{W}_Q^{(m)} = \frac{1}{4\pi} \sum_{n=1}^{N-2} \int_{\psi=0}^{\infty} \Gamma_m(\varphi_n+\psi) \Big[(\chi_n^{(0)})^2 + (r\cos(\varphi_0+\alpha) - s_n^{(0)}\cos(\varphi_n+\psi))^2 +$$

$$+ (r\sin(\varphi_0+\alpha) - s_n^{(0)}\sin(\varphi_n+\psi) - k_x\psi)^2 \Big]^{-3/2} \cdot \Big\{ \Big[(s_n^{(0)})^2 - r s_n^{(0)} \cos\big(\alpha - \psi - \tfrac{2\pi n}{N}\big)$$

$$- k_x r\cos(\varphi_0+\alpha) + k_x s_n^{(0)}\cos(\varphi_n+\psi) + k_x s_n^{(0)} \psi \sin(\varphi_n+\psi) \Big] \mathcal{W}_x +$$

$$+ \Big[(s_n^{(0)}\sin(\varphi_n+\psi) - r\sin(\varphi_0+\alpha) + k_x\psi) \frac{\partial}{\partial\psi} \chi_n^{(0)} - (k_x + s_n^{(0)}\cos(\varphi_n+\psi)) \chi_n^{(0)} \Big] \mathcal{W}_y$$

$$+ \Big[(r\cos(\varphi_0+\alpha) - s_n^{(0)}\cos(\varphi_n+\psi)) \frac{\partial}{\partial\psi} \chi_n^{(0)} - s_n^{(0)}\sin(\varphi_n+\psi) \cdot \chi_n^{(0)} \Big] \mathcal{W}_z \Big\} d\psi$$

$$- \frac{1}{4\pi} \sum_{n=1}^{N-2} \int_{\psi=0}^{\infty} \Gamma_m(\varphi_n+\psi) \big[\text{Integrand with } s_n^{(i)} \big] d\psi \; ;$$

with $\quad \chi_n^{(0)} = \psi\, k_0(s_n^{(0)}, \varphi_n+\psi) \; ; \quad \chi_n^{(i)} = \psi\, k_0(s_n^{(i)}, \varphi_n+\psi) \; .$ (5)

$$\mathcal{W}_Q^{(0)} = \frac{1}{4\pi} \int_{R_i}^{R_0} \int_{0}^{2\alpha} \frac{\partial\Gamma(s,\varphi_0+\psi)}{\partial s} \Big[(r\cos(\varphi_0+\alpha) - s\cos(\varphi_0+\psi))^2 +$$

$$+ (r\sin(\varphi_0+\alpha) - s\sin(\varphi_0+\psi) - k_x\psi)^2 \Big]^{-3/2} \Big[r s\cos(\alpha-\psi) - s^2 +$$

$$+ k_x r\cos(\varphi_0+\alpha) - k_x s\cos(\varphi_0+\psi) - k_x s\psi\sin(\varphi_0+\psi) \Big] d\psi\, ds \cdot \mathcal{W}_x +$$

$$+ \frac{1}{4\pi} \int_{R_i}^{R_0} \int_{2\alpha}^{\infty} \frac{\partial\Gamma(s,\varphi_0+\psi)}{\partial s} \Big[(\chi_0)^2 + (r\cos(\varphi_0+\alpha) - s\cos(\varphi_0+\psi))^2 + \qquad (6)$$

$$+ (r\sin(\varphi_0+\alpha) - s\sin(\varphi_0+\psi) - k_x\psi)^2 \Big]^{-3/2} \Big\{ \Big[r s\cos(\alpha-\psi) - s^2 +$$

$$+ k_x r\cos(\varphi_0+\alpha) - k_x s\cos(\varphi_0+\psi) - k_x s\psi\sin(\varphi_0+\psi) \Big] \mathcal{W}_x +$$

$$+ \Big[(r\sin(\varphi_0+\alpha) - s\sin(\varphi_0+\psi) - k_x\psi) \frac{\partial\chi_0}{\partial\psi} + (k_x + s\cos(\varphi_0+\psi)) \chi_0 \Big] \mathcal{W}_y +$$

$$+ \Big[(s\cos(\varphi_0+\psi) - r\cos(\varphi_0+\alpha)) \frac{\partial\chi_0}{\partial\psi} + s\sin(\varphi_0+\psi) \cdot \chi_0 \Big] \mathcal{W}_z \Big\} d\psi\, ds \; ;$$

with $\quad \chi_0 = (\psi - 2\alpha)\, k_0(s, \varphi_0+\psi) \; .$

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In formula (4) $\psi_o^{(o)}$ and $\psi_o^{(i)}$ indicate the angle by which the tip vortex and root vortex of the n = N - 1 blade intersect the 3/4-line of the n = 0 airfoil space point. We designate the related radii by r_o and r_i. Then the following applies, according to definition (Figures 6 and 8):

$$
\begin{aligned}
r_o \cos(\varphi_o+\alpha_o) &= s_{N-1}^{(o)} \cos\left(\varphi_o - \tfrac{2\pi}{N} + \psi_o^{(o)}\right) ; \qquad \alpha_o = \alpha(r_o) \\
r_o \sin(\varphi_o+\alpha_o) &= s_{N-1}^{(o)} \sin\left(\varphi_o - \tfrac{2\pi}{N} + \psi_o^{(o)}\right) + k_x \psi_o^{(o)} ; \\
r_i \cos(\varphi_o+\alpha_i) &= s_{N-1}^{(i)} \cos\left(\varphi_o - \tfrac{2\pi}{N} + \psi_o^{(i)}\right) ; \qquad \alpha_i = \alpha(r_i) \\
r_i \sin(\varphi_o+\alpha_i) &= s_{N-1}^{(i)} \sin\left(\varphi_o - \tfrac{2\pi}{N} + \psi_o^{(i)}\right) + k_x \psi_o^{(i)} ;
\end{aligned}
\qquad (7)
$$

c) For the free longitudinal vortex we select the following model:

At the n = 0 airfoil the vortices are arranged up to the 1/4 airfoil depth behind the blade in the plane of the rotor (hence up to $\psi = 2\alpha$) (linearized theory). Thereafter they depend on the x = 0 plane with trapezoidal flow-through distribution (equation (1)) (Figure 4).

The longitudinal vortices of the n = 1 to n = N - 2 blades, whose influence on the airflow boundary condition is not so important, are carried away from the plane of the rotor with trapezoidal flow-through in the x-direction.

For the treatment of the longitudinal vortex of the forward-driving n = N - 1 blade to the airfoil space point, two models are discussed in the following:

Case I: Vortex regulation exactly as for n = 1 to n = N-2 airfoils.

Case II: By analogy with the model applicable to transverse vortices, the longitudinal vortices remain up to about 1/4 of the blade depth behind the airfoil space point in the x = 0 plane of the rotor, and are only thereafter blown away with strained flow-though (1) in the x-direction (Figure 6).

With this concept the induced velocity field is derived from the free longitudinal vortex in the 3/4-line of the airfoil space point x = 0, $y = r \cos(\phi_o + \alpha)$, $Z = r \sin(\phi_o + \alpha)$ in the formula:

$$\mathcal{W}_L = \sum_{n=0}^{N-1} \mathcal{W}_L^{(n)}$$

$$\mathcal{W}_L^{(0)} = \frac{1}{4\pi} \int_{R_i}^{R_0} \int_{0}^{2\alpha} \frac{\partial \Gamma(s, \varphi_0+\psi)}{\partial \varphi_0} \Big[(r\cos(\varphi_0+\alpha) - s\cos(\varphi_0+\psi))^2 +$$

$$+ (r\sin(\varphi_0+\alpha) - s\sin(\varphi_0+\psi) - k_* \psi)^2 \Big]^{-3/2} \cdot \Big[r\sin(\alpha-\psi) - k_* \psi \cos(\varphi_0+\psi) \Big] \cdot$$

$$\cdot d\psi\, ds \cdot \mathcal{W}_x + \frac{1}{4\pi} \int_{R_i}^{R_0} \int_{2\alpha}^{\infty} \frac{\partial \Gamma(s, \varphi_0+\psi)}{\partial \varphi_0} \Big[(\chi_0)^2 + (r\cos(\varphi_0+\alpha) - s\cos(\varphi_0+\psi))^2 +$$

$$+ (r\sin(\varphi_0+\alpha) - s\sin(\varphi_0+\psi) - k_* \psi)^2 \Big]^{-3/2} \cdot \Big\{ \Big[r\sin(\alpha-\psi) - k_* \psi \cos(\varphi_0+\psi) \Big] \mathcal{W}_x$$

$$+ \Big[(s\sin(\varphi_0+\psi) - r\sin(\varphi_0+\alpha) + k_* \psi) \frac{\partial \chi_0}{\partial s} - \sin(\varphi_0+\psi) \cdot \chi_0 \Big] \mathcal{W}_y + \qquad (8)$$

$$+ \Big[(r\cos(\varphi_0+\alpha) - s\cos(\varphi_0+\psi)) \frac{\partial \chi_0}{\partial s} + \cos(\varphi_0+\psi) \cdot \chi_0 \Big] \mathcal{W}_z \Big\}\, d\psi\, ds \;;$$

with $\chi_0 = (\psi - 2\alpha)\, k_0(s, \varphi_0+\psi)$.

$$\sum_{n=1}^{N-2} \mathcal{W}_L^{(n)} = \frac{1}{4\pi} \sum_{n=1}^{N-2} \int_{R_i}^{R_0} \int_{0}^{\infty} \frac{\partial \Gamma(s, \varphi_n+\psi)}{\partial \varphi_0} \Big[(\chi_n)^2 + (r\cos(\varphi_0+\alpha) - s\cos(\varphi_n+\psi))^2 +$$

$$+ (r\sin(\varphi_0+\alpha) - s\sin(\varphi_n+\psi) - k_* \psi)^2 \Big]^{-3/2} \cdot \Big\{ \Big[r\sin(\alpha-\psi-\tfrac{2\pi n}{N}) -$$

$$- k_* \psi \cos(\varphi_n+\psi) \Big] \mathcal{W}_x + \Big[(s\sin(\varphi_n+\psi) - r\sin(\varphi_0+\alpha) + k_* \psi) \frac{\partial \chi_n}{\partial s} - \qquad (9)$$

$$- \sin(\varphi_n+\psi) \cdot \chi_n \Big] \mathcal{W}_y + \Big[(r\cos(\varphi_0+\alpha) - s\cos(\varphi_n+\psi)) \frac{\partial \chi_n}{\partial s} +$$

$$+ \cos(\varphi_n+\psi) \cdot \chi_n \Big] \mathcal{W}_z \Big\}\, d\psi\, ds \;;$$

with $\chi_n = \psi\, k_0(s, \varphi_n+\psi)$.

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$$w_L^{(N-1)} = \frac{1}{4\pi}\int_{R_i}^{R_0}\int_{0}^{\alpha+\Psi_{N-1}} \frac{\partial\Gamma(s,\varphi_0-\frac{2\pi}{N}+\psi)}{\partial\varphi_0}\Big[\big(r\cos(\varphi_0+\alpha)-s\cos(\varphi_0-\tfrac{2\pi}{N}+\psi)\big)^2+$$

$$+\big(r\sin(\varphi_0+\alpha)-s\sin(\varphi_0-\tfrac{2\pi}{N}+\psi)-k_*\psi\big)^2\Big]^{-3/2}\cdot\Big[r\sin(\alpha+\tfrac{2\pi}{N}-\psi)-$$

$$-k_*\psi\cos(\varphi_0-\tfrac{2\pi}{N}+\psi)\Big]\,d\psi\,ds\cdot w_x \;+\; \frac{1}{4\pi}\int_{R_i}^{R_0}\int_{\alpha+\Psi_{N-1}}^{\infty}\frac{\partial\Gamma(s,\varphi_0-\frac{2\pi}{N}+\psi)}{\partial\varphi_0}\cdot$$

$$\cdot\Big[(x_{N-1})^2+\big(r\cos(\varphi_0+\alpha)-s\cos(\varphi_0-\tfrac{2\pi}{N}+\psi)\big)^2+\big(r\sin(\varphi_0+\alpha)- \qquad (10)$$

$$-s\sin(\varphi_0-\tfrac{2\pi}{N}+\psi)-k_*\psi\big)^2\Big]^{-3/2}\cdot\Big\{\Big[r\sin(\alpha+\tfrac{2\pi}{N}-\psi)-$$

$$-k_*\psi\cos(\varphi_0-\tfrac{2\pi}{N}+\psi)\Big]w_x+\Big[\big(s\sin(\varphi_0-\tfrac{2\pi}{N}+\psi)-r\sin(\varphi_0+\alpha)+k_*\psi\big)\frac{\partial x_{N-1}}{\partial s}$$

$$-\sin(\varphi_0-\tfrac{2\pi}{N}+\psi)\cdot x_{N-1}\Big]w_y+\Big[\big(r\cos(\varphi_0+\alpha)-s\cos(\varphi_0-\tfrac{2\pi}{N}+\psi)\big)\frac{\partial x_{N-1}}{\partial s}+$$

$$+\cos(\varphi_0-\tfrac{2\pi}{N}+\psi)\cdot x_{N-1}\Big]w_z\Big\}\,d\psi\,ds;$$

with $\quad x_{N-1}=(\psi-\alpha-\Psi_{N-1})\,k_0(s,\varphi_0-\frac{2\pi}{N}+\psi)\cdot$ $\quad$ Case II.

In Case I $L^{(N-1)}$ is to be taken directly from formula (9). The angle Ψ_{N-1} in equation (10) is defined by the singularity of the integrand of the first integral. The latter becomes exact, then singular, at $\psi = \Psi_{N-1}$ and $S = S^*_{N-1}$. It is valid, therefore, according to the definition: /8

$$\begin{aligned} r\cos(\varphi_0+\alpha) &= s^*_{N-1}\cos(\varphi_0-\tfrac{2\pi}{N}+\Psi_{N-1})\,; \\ r\sin(\varphi_0+\alpha) &= s^*_{N-1}\sin(\varphi_0-\tfrac{2\pi}{N}+\Psi_{N-1})+k_*\Psi_{N-1}\cdot \end{aligned} \qquad (11)$$

The induced velocities from the vortex system of the rotors at the airfoil space points are thereby indicated to be necessary for the airflow boundary condition.

As usual, we let β be the angle of attack of the blade and R_β the distance of the flapping hinge from the rotor axis. $\omega(r - R_\beta)\, d\beta/d\phi_o$ is therefore the striking velocity. Furthermore, δ_o is the angle of inclination of the skeletal sideview line in its 3/4 points opposite the $r\phi$ axis (Figure 7).

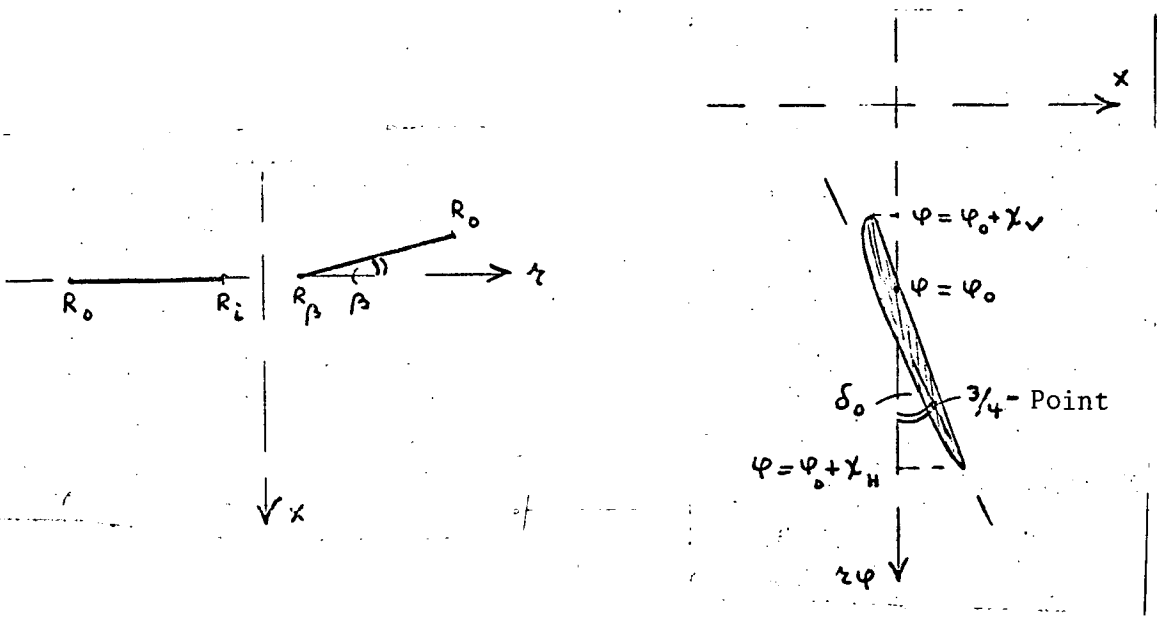

Figure 7

a) Definition of the Angle of Attack $R_\beta = R_i$ is Assumed.

b) Airfoil Section r = Constant.

The airflow boundary condition is stated as follows (Figure 3):

$$[\omega r + W_o \cos(\varphi_o + \alpha)]\, tg\, \delta_o(r, \varphi_o) - u_o + (r - R_\beta)\, \omega \frac{d\beta}{d\varphi_o} =$$
$$= u_\pi + u_L + u_Q - (v_\pi + v_L + v_Q)\, tg\, \delta_o(r, \varphi_o)\,. \qquad (12)$$
$$(R_i \le r \le R_o;\quad 0 \le \varphi_o \le 2\pi)\quad (v = -v \sin(\varphi_o + \alpha) + W \cos(\varphi_o + \alpha))\,.$$

The relationship (12) yields, after insertion of the induced velocity, an integral equation for calculating the airfoil circulation.

For the solution of equation (12) both flow-through constants k_{oo} and k_{ol} must be determined by an approximate calculation according to one of the known simple theories or, in another way, be fixed on the basis of a prior estimate. After solution of the integral equation and determination of the induced velocity, the utilized $k_{\overline{oo}}$ and $k_{\overline{ol}}$ values are to be checked, and in case it is necessary, be re-determined in order to correct them.

3. Investigation and Transformation of the Singular Core Part of the Integral Equation for the Boundary Condition

The representations (3) to (6) and (8) to (10) of the induced velocity already include the expression which appear as cores of the integral equations, according to boundary conditions (12), so that a repeated notation can be avoided.

We ourselves wish to obtain, above all, a review of the structure of the core portion in order to thereafter develop a suitably devised solution theory. The investigation is therefore to be carried out with special attention to what influence this fact exerts on the circulation distribution, so that the observed airfoil penetrates the unrolled free transverse vortex of the forward-driving blades. Mainly, the bound vortices according to equation (3), as well as the terms of the addend, yield. /9

$$\sum_{n=1}^{N-2} w_Q^{(n)} \quad , \quad \sum_{n=1}^{N-2} w_L^{(n)}$$

according to formulas (5) and (9), only constant core portions in the integral equation, and need to be discussed no further.

For the velocities $\omega_Q^{(o)}$, $\omega_Q^{(n-1)}$, $\omega_L^{(o)}$, $\omega_L^{(n-1)}$ the integrands of the second integral are sometimes constant. This follows both from formulas (4), (6), (8) and (10) and also directly from the physical concept of the applied vortex models.

A thorough investigation consequently requires the first integral component of the final stated velocities. It includes, besides, only one x-component. These are indicated by us with the term $(u_Q^{(o)})$, etc.

a) We examine the airfoil space point n = 0.

The denominator of the term $(u_Q^{(o)} + u_L^{(o)})$ disappears in favor of the suggested space point coordinates r and $\phi_o + \alpha(r)$ when $S = S_o^*$ and $\psi = \Psi_o$; thereby S_o^* and Ψ_o are determined by the following equations:

$$\left.\begin{aligned} r\cos(\varphi_o+\alpha) &= s_o^*\cos(\varphi_o+\Psi_o) \\ r\sin(\varphi_o+\alpha) &= s_o^*\sin(\varphi_o+\Psi_o) + k_*\Psi_o \,. \end{aligned}\right| \qquad (13)$$

We expand the numerator and the denominator of the integrand from the term $(u_Q^{(o)} + u_L^{(o)})$ in the vicinity of the locus (13). With the substitution of

$$\psi = \Psi_o + \chi \,, \qquad s = s_o^* + \sigma$$

there is obtained, after simple conversions, provided an additional third-order in x term in the denominator and an additional second-order in σ term in the numerator are taken into consideration.

$$Ne = \left[\sigma^2 + 2\sigma\chi k_*\sin(\varphi_o+\Psi_o) + \chi^2(s_o^{*2} + k_*^2 + 2k_* s_o^*\cos(\varphi_o+\Psi_o) - \right. \qquad (14)$$
$$\left. - \chi^3 s_o^* k_*\sin(\varphi_o+\Psi_o) + \sigma\chi^2(s_o^* + 2k_*\cos(\varphi_o+\Psi_o))\right]^{3/2} ;$$

and equally analogous with the utilization of the relationship (13):

/10

$$\left.\begin{aligned} 2\ddot{a} = &- \frac{\partial\Gamma(s,\varphi_o+\Psi_o+\chi)}{\partial s}\left[\sigma(s_o^* + k_*\cos(\varphi_o+\Psi_o)) + \sigma^2 + \right. \\ &+ \tfrac{1}{2}\chi^2 s_o^*(s_o^* + k_*\cos(\varphi_o+\Psi_o))\Big] - \frac{\partial\Gamma(s,\varphi_o+\Psi_o+\chi)}{\partial\varphi_o}\left[\chi(s_o^* + k_*\cos(\varphi_o+\Psi_o))\right. \\ &\left. - \chi^2 k_*\sin(\varphi_o+\Psi_o)\right] . \end{aligned}\right| \qquad (15)$$

We now introduce the following shortened expression:

$$A_o = s_o^{*2} + k_*^2 + 2k_* s_o^* \cos(\varphi_o + \Psi_o) \; ; \quad B_o = k_* \sin(\varphi_o + \Psi_o) \; ; \quad \Delta_o^2 = (s_o^* + k_* \cos(\varphi_o + \Psi_o))^2 = A_o - B_o^2 \; . \tag{16}$$

In order to obtain a general idea of the values of S_o^* and Ψ_o in the solution of the equation system (13), we have to bear in mind that the limiting case where $k_* \to 0$

$$s_o^* = r \; ; \quad \Psi_o = \alpha \; .$$

holds.

If we so restrict ourselves to small degrees of advance, as k_*/R_o, as is usual in the line-support theory of the rotor, so that no backflow appears, then for all blade space points $r + k_* \cos(\phi_o + \alpha) > 0$ applies. If we reflect further that $\alpha^2 << 1$ is also true, then it appears applicable to solve equation (13) in the linearized form. We obtain:

$$s_o^* \approx r - \frac{k_* r \alpha \sin(\varphi_o + \alpha)}{r + k_* \cos(\varphi_o + \alpha)} \; ; \quad \Psi_o \approx \frac{r\alpha}{r + k_* \cos(\varphi_o + \alpha)} \; . \tag{18}$$

From (18) it then follows that

$$\Delta_o = r + k_* \cos(\phi_o + \alpha) - k_* \sin(\Phi_o + \alpha) + (k_{*1}^2 \alpha^2) \text{ term.}$$

If $r + k_* \cos(\phi_0 + \alpha) > 0$ thereby applies, $\Delta_o > 0$ also remains.

Due to the zero position $S = S_o^*$, $\psi = \Psi_o$ of the $n = 0$ core position in the demoninator, the hypothetical singularity depends upon the momentary airfoil position ϕ_o. There are r- and ϕ_o- values present throughout[3], by which $S_o^* > R$ or $S_o^* < R_i$ occur, and consequently the singularity for short time ranges

---

[3] At $\phi_o + \alpha \approx 3\pi/2$ and $r < R_o$ as well as at $\phi_o + \alpha \approx \pi/2$ and $r > R_i$.

(therefore ϕ_o-interval) drops off. Then s and r can really assume values between R_i and R_o.

The solution theory from integral equations becomes very complicated and costly with core singularities of that kind, especially when singularities of such a nature appear repeatedly and are of a different type.

As it will turn out, such singularities are also contained in the core portions of the n = N-1 airfoil. Here is a precise consideration more significant than the n = 0 airfoil, since the value of $k_* \Psi_{N-1}$ from equation (11) is considerably larger than that of $k_* \Psi_o$ from equations (13) or (18).

It appears applicable therefrom to carry out with the replaceable approximate postulate $k_* = 0$ the analysis of the portion, the $(u_Q^{(o)} + u_L^{(o)})$ term, at a small k_* without backflow of the blade. The core singularity is then given in the simpler form (17). We obtain from (6) and (8) with (14) and (15) the clear construction:

$$
\begin{aligned}
\text{Term}\{u_Q^{(o)} + u_L^{(o)}\} = \frac{1}{4\pi} \int_{R_i}^{R_o} \int_{-\alpha}^{\alpha} \Big\{ \Big[r^2 + s^2 - 2rs\cos\chi \Big]^{-3/2} \Big[\frac{\partial \Gamma(s, \varphi_o + \alpha + \chi)}{\partial s} \cdot \\
\cdot (rs\cos\chi - s^2) - \frac{\partial \Gamma(s, \varphi_o + \alpha + \chi)}{\partial \varphi_o} r \sin\chi \Big] - \Big[(r-s)^2 + r^2\chi^2 \Big]^{-3/2} \cdot \\
\cdot \Big(1 + \frac{3}{2} \frac{\chi^2 r (r-s)}{(r-s)^2 + r^2\chi^2} \Big) \Big[\frac{\partial \Gamma(s, \varphi_o + \alpha + \chi)}{\partial s} \Big(r(r-s) - (r-s)^2 - \frac{1}{2} r^2\chi^2 \Big) - \\
- \frac{\partial \Gamma(s, \varphi_o + \alpha + \chi)}{\partial \varphi_o} r\chi \Big] \Big\} d\chi\, ds + \frac{1}{4\pi} \int_{R_i}^{R_o} \Big\{ \int_{-\alpha}^{\alpha} \Big[(r-s)^2 + r^2\chi^2 \Big]^{-3/2} \cdot \\
\cdot \Big(1 + \frac{3}{2} \frac{\chi^2 r (r-s)}{(r-s)^2 + r^2\chi^2} \Big) \Big[\frac{\partial \Gamma(s, \varphi_o + \alpha + \chi)}{\partial s} \Big(r(r-s) - (r-s)^2 - \frac{1}{2} r^2\chi^2 \Big) - \\
- \frac{\partial \Gamma(s, \varphi_o + \alpha + \chi)}{\partial \varphi_o} r\chi \Big] d\chi - \Big(\frac{2}{r-s} + \frac{1}{r} \ln \frac{|r-s|}{2r\alpha} \Big) \frac{\partial \Gamma(s, \varphi_o + \alpha)}{\partial s} - \\
- \frac{2}{r^2} \ln \frac{|r-s|}{2r\alpha} \frac{\partial^2 \Gamma(s, \varphi_o + \alpha)}{\partial \varphi_o^2} \Big\} ds - \frac{1}{4\pi} \int_{R_i}^{R_o} \Gamma(s, \varphi_o + \alpha) \frac{ds}{rs} + \\
+ \frac{1}{2\pi} \int_{R_i}^{R_o} \frac{\partial^3 \Gamma(s, \varphi_o + \alpha)}{\partial s \partial \varphi_o^2} (r-s) \Big[\ln \frac{|r-s|}{2r\alpha} - 1 \Big] \frac{ds}{r^2} + \\
+ \frac{1}{2\pi} \int_{R_i}^{R_o} \Big(\frac{\partial \Gamma(s, \varphi_o + \alpha)}{\partial s} + \frac{1}{2s} \Gamma(s, \varphi_o + \alpha) \Big) \frac{ds}{r - s} \cdot
\end{aligned}
\tag{19}
$$

In formula (19) the integrand of the first double integral is constant, and disappears at the point where x = 0 and s = r. With the use of the integration formulas (74) to (81) from No. 7, it is shown that the integrand of the second integral over s for s = r takes on the value:

$$\frac{1}{4\pi}\left(\frac{1}{2r}\frac{\partial\Gamma(r,\varphi_0+\alpha)}{\partial r}+\frac{2}{r^2}\frac{\partial^2\Gamma(r,\varphi_0+\alpha)}{\partial\varphi_0^2}\right)+O(\alpha^2)$$

which, therefore, is also constant, The remaining terms become partly transformed through partial integration, so that the singular core portion in the last line of (19) stands out distinctly. For this reason

$$\Gamma(R_i,\varphi_0)=\Gamma(R_a,\varphi_0)=0 \qquad (20)$$

is assumed. (See No. 7, formulas (82), (83)).

b) Forward-driving n = N-1 airfoil, longitudinal vortex. /12

We examine Case II according to equation (10), since in Case I the integrand is constant anyway.

The integrand from the $(u_L^{(N-1)}$ term in Case II becomes singular at the simulated space point r, $\phi + \alpha(r)$ when the equation system (11) corresponds to $S = S^*_{N-1}$ and $\psi = \Psi_{N-1}$. In the neighborhood of this position the denominator takes the form of (14) from the $(u_L^{(N-1)}$ term. S^*_o is to be replaced only by S^*_{N-1}, Ψ_o by Ψ_{N-1} and $(\phi + \Psi_o)$ by $(\phi_o - \frac{2\pi}{N} + \Psi_{N-1})$. Similarly there is yielded for the numerator:

$$\text{Numerator} = -\frac{\partial\Gamma(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}+\chi)}{\partial\varphi_0}\left[\chi\left(s^*_{N-1}+k_*\cos(\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}\right)-\right.$$
$$\left.-\chi^2 k_*\sin(\varphi_0-\frac{2\pi}{N}-\Psi_{N-1})\right]. \qquad (21)$$

If we assume that in the considered space point r, $\phi_o + \alpha$, no limited anomalous increase through vortex deformation of the free vortex element is present, then the values $S = S^*_{N-1}$ and $\psi = \Psi_{N-1}$ describe an isolated singularity in the denominator, for they must eliminate both σ and also χ, ($\psi = \Psi_{N-1} + \chi$;

$S = S^*_{N-1} + \sigma$). Since the principal part of the denominator for small values of σ and χ has the formula

$$[S^*_{N-1}\sin(\varphi_0 - \tfrac{2\pi}{N} + \Psi_{N-1})\cdot\chi - \cos(\varphi_0 - \tfrac{2\pi}{N} + \Psi_{N-1})\cdot\sigma]^2 +$$
$$+ [S^*_{N-1}\cos(\varphi_0 - \tfrac{2\pi}{N} + \Psi_{N-1})\cdot\chi + \sin(\varphi_0 - \tfrac{2\pi}{N} + \Psi_{N-1})\cdot\sigma + k_*\chi]^2$$

the determinant must be

$$S^*_{N-1} + k_*\cos(\varphi_0 - \tfrac{2\pi}{N} + \Psi_{N-1}) > 0 \qquad (22)$$

provided the zero position $\sigma = 0$ and $\chi = 0$ should be possible. The relationship (22) is essential for the further transformation.

One can begin solving for small k_* value (i.e., without backflow from space points) in the equation system (11) using a linearized approximation, if products of the type $(S^*_{N-1} - r)(\Psi_{N-1} - \frac{2\pi}{N} - \alpha)$ are neglected to begin with. There is thus obtained:

$$S^*_{N-1} \approx 2 - \frac{k_* 2(\frac{2\pi}{N} + \alpha)\sin(\varphi_0 + \alpha)}{2 + k_*\cos(\varphi_0 + \alpha)} \quad ; \quad \Psi_{N-1} \approx \frac{2\pi}{N} + \alpha - \frac{k_*(\frac{2\pi}{N} + \alpha)\cos(\varphi_0 + \alpha)}{2 + k_*\cos(\varphi_0 + \alpha)} \qquad (23)$$

The approximate solution in (23) is naturally less accurate than the corresponding relationship (18) for the n = 0 airfoil. Nevertheless (23) is repeatedly improved. As a consequence of $(\alpha + \frac{2\pi}{N})^2 >> \alpha^2$ the angle Ψ_{N-1} is substantially larger than Ψ_0, then K_* also need not be neglected or set at zero (other than at the n = 0 airfoil) by further conversion from the $(u_L^{(N-1)})$ term.

With the reduced terms /13

$$A_{N-1} = S^{*2}_{N-1} + k_*^2 + 2k_* S^*_{N-1}\cos(\varphi_0 - \tfrac{2\pi}{N} + \Psi_{N-1})$$
$$B_{N-1} = k_*\sin(\varphi_0 - \tfrac{2\pi}{N} + \Psi_{N-1}) \quad ; \quad \Delta^2_{N-1} = (S^*_{N-1} + k_*\cos(\varphi_0 - \tfrac{2\pi}{N} + \Psi_{N-1}))^2 = A_{N-1} - B^2_{N-1}$$

we obtain by analogy with formula (19) the following expression:

$$\text{Term}\,|\,(u_L^{(N-1)}) = \frac{1}{4\pi}\int_{R_i}^{R_0}\int_{-\Psi_{N-1}}^{-\alpha}\Big[\big(2\cos(\varphi_0+\alpha) - s\cos(\varphi_0-\tfrac{2\pi}{N}+\Psi_{N-1}+\chi)\big)^2 +$$

$$+\big(2\sin(\varphi_0+\alpha) - s\sin(\varphi_0-\tfrac{2\pi}{N}+\Psi_{N-1}+\chi) - k_x(\Psi_{N-1}+\chi)\big)^2\Big]^{-3/2}\cdot$$

$$\cdot\Big[2\sin(\varphi_0+\alpha)\cos(\varphi_0-\tfrac{2\pi}{N}+\Psi_{N-1}+\chi) - 2\cos(\varphi_0+\alpha)\sin(\varphi_0-\tfrac{2\pi}{N}+\Psi_{N-1}+\chi) -$$

$$-k_x(\Psi_{N-1}+\chi)\cos(\varphi_0-\tfrac{2\pi}{N}+\Psi_{N-1}+\chi)\Big]\frac{\partial\Gamma(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}+\chi)}{\partial\varphi_0}\,d\chi\,ds -$$

$$-\frac{1}{4\pi}\int_{R_i}^{R_0}\Big\{\int_{-\alpha}^{\alpha}\Big[\sigma^2+2\sigma\chi B_{N-1}+\chi^2A_{N-1}\Big]^{-3/2}\cdot\Big(1+\frac{3}{2}\,\frac{\chi^3 s_{N-1}B_{N-1}+\sigma\chi^2(s_{N-1}-2\Delta_{N-1})}{\sigma^2+2\sigma\chi B_{N-1}+\chi^2A_{N-1}}\Big)\cdot$$

$$\cdot(\chi\Delta_{N-1}-\chi^2B_{N-1})\frac{\partial\Gamma(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1}+\chi)}{\partial\varphi_0}\,d\chi + \frac{2B_{N-1}}{\sqrt{A_{N-1}}}\Big(\frac{1}{\Delta_{N-1}}\,\frac{1}{\sigma} -$$

$$-\frac{1}{A_{N-1}}\Big(1-\frac{3}{2}\,\frac{s_{N-1}\Delta_{N-1}}{A_{N-1}}\Big)\ln\frac{|\sigma|\Delta_{N-1}}{2\alpha A_{N-1}}\Big)\frac{\partial\Gamma(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1})}{\partial\varphi_0} + \tag{24}$$

$$+\frac{2\Delta_{N-1}}{\sqrt{A_{N-1}}^3}\ln\frac{|\sigma|\Delta_{N-1}}{2\alpha A_{N-1}}\cdot\frac{\partial^2\Gamma(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1})}{\partial\varphi_0^2}\Big\}ds +$$

$$+\frac{1}{2\pi}\int_{R_i}^{R_0}\Big\{\frac{B_{N-1}}{\sqrt{A_{N-1}}^3}\Big(1-\frac{3}{2}\,\frac{s_{N-1}\Delta_{N-1}}{A_{N-1}}\Big)\frac{\partial^2\Gamma(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1})}{\partial\varphi_0\partial s} -$$

$$-\frac{\Delta_{N-1}}{\sqrt{A_{N-1}}^3}\,\frac{\partial^3\Gamma(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1})}{\partial\varphi_0^2\partial s}\Big\}\Big[\ln\frac{|\sigma|\Delta_{N-1}}{2\alpha A_{N-1}}-1\Big]\sigma\,ds -$$

$$-\frac{1}{2\pi}\,\frac{B_{N-1}}{\sqrt{A_{N-1}}}\,\frac{1}{\Delta_{N-1}}\int_{R_i}^{R_0}\frac{\partial\Gamma(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1})}{\partial\varphi_0}\,\frac{ds}{s_{N-1}^*-s}\quad;\quad(\sigma=s-s_{N-1}^*).$$

In formula (24) the first double integral has a constant integrand. With the use of the integral formulas (74) to (81) from No. 7, the integrand of the second integral over s for $S = S_{N-1}^*$ yields, furthermore, the value

$$+\frac{1}{2\pi}\,\frac{\partial\Gamma(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1})}{\partial\varphi_0}\Big[\frac{3B_{N-1}^3-4A_{N-1}B_{N-1}}{\Delta_{N-1}^2\sqrt{A_{N-1}}^3}+\frac{s_{N-1}^*B_{N-1}(3{,}5A_{N-1}-4B_{N-1}^2)}{\Delta_{N-1}\sqrt{A_{N-1}}^5}\Big]$$

$$-\frac{1}{2\pi}\,\frac{\partial^2\Gamma(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1})}{\partial\varphi_0^2}\,\frac{A_{N-1}-2B_{N-1}^2}{\Delta_{N-1}\sqrt{A_{N-1}}^3}+O(\alpha^2)\,.$$

The last term in formula (24) contains the sole singular core portion (see also formulas (82), (83), No. 7).

c) Forward-driving n = N-1 airfoil; unrolled transverse vortex. /14

The velocity component, the $(u_Q^{(N-1)})$ term, for investigation exists from the unrolled tip vortex and root vortex. Since both are built up quite analogously, we carry out the discussion only for one of the integrands and omit the higher indices (o) and (i). The singularity in the integrand of the $(u_Q^{(N-1)})$ term is given by the equation system (7). From this the values of r_o and ψ_o are to be calculated by the suggested S_{N-1} and ϕ_o.

We develop the denominator and numerator from the $(u_Q^{(N-1)})$ term in the in the vicinity of this locus, thus setting up:

$$\psi = \psi_o + \chi \,, \quad r = r_o + \varrho \,, \quad \alpha(r) = \alpha_o + \alpha_o' \varrho$$

and take into consideration in the denominator additional third order terms χ, in the numerator only second order terms in Q. From consideration of (7), after a few simple conversions there results:

$$\begin{aligned}
\text{Denominator} = \Big\{ & \varrho^2 (1 + r_o^2 \alpha_o'^2) + 2\varrho\chi \left[S_{N-1} \sin\left(\psi_o - \tfrac{2\pi}{N} - \alpha_o\right) - k_* \sin(\varphi_o + \alpha_o) - \right. \\
& - S_{N-1} r_o \alpha_o' \cos\left(\psi_o - \tfrac{2\pi}{N} - \alpha_o\right) - k_* r_o \alpha_o' \cos(\varphi_o + \alpha_o) \Big] + \chi^2 \left[S_{N-1}^2 + k_*^2 + \right. \\
& \left. + 2 k_* S_{N-1} \cos\left(\varphi_o - \tfrac{2\pi}{N} + \psi_o\right) \right] - \chi^3 k_* S_{N-1} \sin\left(\varphi_o - \tfrac{2\pi}{N} + \psi_o\right) + \varrho^3 r_o \alpha_o'^2 + \\
& + \varrho \chi^2 S_{N-1} \left[\cos\left(\psi_o - \tfrac{2\pi}{N} - \alpha_o\right) + r_o \alpha_o' \sin\left(\psi_o - \tfrac{2\pi}{N} - \alpha_o\right) \right] - \\
& - 2\varrho^2 \chi \alpha_o' \left[S_{N-1} \cos\left(\psi_o - \tfrac{2\pi}{N} - \alpha_o\right) + k_* \cos(\varphi_o + \alpha_o) + S_{N-1} \tfrac{r_o}{2} \alpha_o' \sin\left(\psi_o - \tfrac{2\pi}{N} - \alpha_o\right) - \right. \\
& \left. - k_* \tfrac{r_o}{2} \alpha_o' \sin(\varphi_o + \alpha) \right] \Big\}^{3/2} \,;
\end{aligned} \tag{25}$$

and neglecting Γ_m (indicating tip vortex):

$$\begin{aligned}
\text{Numerator} = & - \varrho \left[S_{N-1} \cos\left(\psi_o - \tfrac{2\pi}{N} - \alpha_o\right) + k_* \cos(\varphi_o + \alpha_o) \right] + \varrho\chi S_{N-1} \sin\left(\psi_o - \tfrac{2\pi}{N} - \alpha_o\right) + \\
& + \tfrac{1}{2} \chi^2 S_{N-1} \left[S_{N-1} + k_* \cos\left(\varphi_o - \tfrac{2\pi}{N} + \psi_o\right) \right] - \alpha_o' \varrho (r_o + \varrho) \left[S_{N-1} \sin\left(\psi_o - \tfrac{2\pi}{N} - \alpha_o\right) - \right. \\
& \left. - k_* \sin(\varphi_o + \alpha) \right] + \tfrac{1}{2} \varrho^2 r_o \alpha_o'^2 \left[S_{N-1} \cos\left(\psi_o - \tfrac{2\pi}{N} - \alpha_o\right) + k_* \cos(\varphi_o + \alpha_o) \right] - \\
& - \varrho\chi S_{N-1} r_o \alpha_o' \cos\left(\psi_o - \alpha_o - \tfrac{2\pi}{N}\right) .
\end{aligned} \tag{26}$$

If we limit ourselves as hitherto to small k_* values (i.e., no back airflow), thus $S_{N-1} + k_* \cos(\phi_0 + \alpha_0) > 0$, the equation system (7) can therefore be linearized. Its approximate solution then is as follows:

$$r_0 \approx S_{N-1} + \frac{k_* (\alpha_0 + \frac{2\pi}{N}) S_{N-1} \sin(\varphi_0 + \alpha_0)}{S_{N-1} + k_* \cos(\varphi_0 + \alpha_0)} \; ; \quad \psi_0 \approx \frac{(\alpha_0 + \frac{2\pi}{N}) S_{N-1}}{S_{N-1} + k_* \cos(\varphi_0 + \alpha_0)} \; . \tag{27}$$

We introduce the following reduction of terms: /15

$$\begin{aligned}
A &= S_{N-1}^2 + k_*^2 + 2 k_* S_{N-1} \cos(\varphi_0 - \tfrac{2\pi}{N} + \psi_0) \; ; \quad C = (1 + r_0^2 \alpha_0'^2) \; ; \\
B &= S_{N-1} \sin(\psi_0 - \tfrac{2\pi}{N} - \alpha_0) - k_* \sin(\varphi_0 + \alpha_0) - S_{N-1} r_0 \alpha_0' \cos(\psi_0 - \tfrac{2\pi}{N} - \alpha_0) - \\
&\quad - k_* r_0 \alpha_0' \cos(\varphi_0 + \alpha_0) \; ; \qquad (28) \\
AC - B^2 &= [S_{N-1} \cos(\psi_0 - \tfrac{2\pi}{N} - \alpha_0) + k_* \cos(\varphi_0 + \alpha_0) + \\
&\quad + S_{N-1} r_0 \alpha_0' \sin(\psi_0 - \tfrac{2\pi}{N} - \alpha_0) - k_* r_0 \alpha_0' \sin(\varphi_0 + \alpha_0)]^2 \; ; \\
P_1 &= S_{N-1} \cos(\psi_0 - \tfrac{2\pi}{N} - \alpha_0) + k_* \cos(\varphi_0 + \alpha_0) \; ; \\
P_2 &= S_{N-1} \sin(\psi_0 - \tfrac{2\pi}{N} - \alpha_0) - k_* \sin(\varphi_0 + \alpha_0) \; ; \\
P_3 &= \cos(\psi_0 - \tfrac{2\pi}{N} - \alpha_0) + r_0 \alpha_0' \sin(\psi_0 - \tfrac{2\pi}{N} - \alpha_0) \; ; \\
P_4 &= \sin(\psi_0 - \tfrac{2\pi}{N} - \alpha_0) - r_0 \alpha_0' \cos(\psi_0 - \tfrac{2\pi}{N} - \alpha_0) \; .
\end{aligned}$$

The value $AC - B^2$ always varies from zero, being positive. This means, without any physical restriction of the maintained assumption, that the unrolled transverse vortex of the n = N-1 airfoil does not touch, but cuts the 3/4-line of the airfoil space point (Figure 8). Then the direction vector of the unrolled transverse vortex is given by

$$d\delta_Q = [-S_{N-1} \sin(\varphi_0 + \psi - \tfrac{2\pi}{N}) w_y + S_{N-1} \cos(\varphi_0 + \psi - \tfrac{2\pi}{N}) w_z + k_* w_z] \, d\psi \; ,$$

and that of the 3/4-line at the n = 0 airfoil by

$$d\delta_\Gamma = [\cos(\varphi_0 + \alpha) w_y - r \alpha' \sin(\varphi_0 + \alpha) w_y + \sin(\varphi_0 + \alpha) w_z + r \alpha' \cos(\varphi_0 + \alpha) w_z] \, dr \; ,$$

with $\alpha' = \frac{d\alpha}{dr}$.

As is directly verified, the following is valid:

$$(d\sigma_\Gamma \times d\sigma_Q)^2 = AC - B^2 ,$$

and from this the foregoing statement follows.

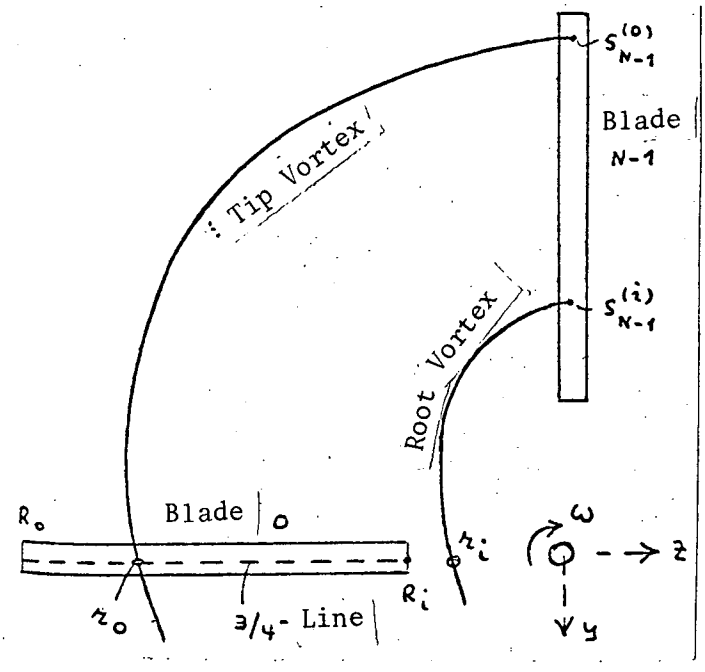

Figure 8. Geometry of the Tip Vortex and Root Vortex of N-1 Blade on Airfoil Space Point.

Through consideration of formulas (25), (26), and (28) we now obtain the sought for expression:

$$\text{Term}\,(u_Q^{(N-1)}) = \frac{1}{4\pi}\int_{-\psi_0}^{-\alpha_0} \Gamma_m(\varphi_0 - \tfrac{2\pi}{N} + \psi_0 + \chi)\Big[\big(r\cos(\varphi_0+\alpha) - S_{N-1}\cos(\varphi_0 - \tfrac{2\pi}{N} + \psi_0 + \chi)\big)^2 +$$
$$+ \big(r\sin(\varphi_0+\alpha) - S_{N-1}\sin(\varphi_0 - \tfrac{2\pi}{N} + \psi_0 + \chi) - k_*(\psi_0+\chi)\big)^2\Big]^{-3/2}\Big[S_{N-1}^2 - \qquad (29)$$
$$- r S_{N-1}\cos(\psi_0+\chi - \tfrac{2\pi}{N} - \alpha) - k_* r\cos(\varphi_0+\alpha) + k_* S_{N-1}\cos(\varphi_0 - \tfrac{2\pi}{N} + \psi_0 + \chi) +$$
$$+ k_* S_{N-1}(\psi_0+\chi)\sin(\varphi_0 - \tfrac{2\pi}{N} + \psi_0 + \chi)\Big]\,d\chi +$$
$$+ \Big\{\frac{1}{4\pi}\int_{-\alpha_0}^{\alpha_0} \Gamma_m(\varphi_0 - \tfrac{2\pi}{N} + \psi_0 + \chi)\big[A\chi^2 + 2B\chi\varsigma + C\varsigma^2\big]^{-3/2}\Big[1 +$$

$$+\frac{3}{2}\frac{\chi^3 k_* S_{N-1}\sin(\varphi_0-\frac{2\pi}{N}+\psi_0)-\varrho^3 r_0\alpha_0'^2-\varrho\chi^2 S_{N-1}P_3+2\varrho^2\chi\alpha_0'(P_1+\frac{r_0}{2}\alpha_0'P_2)}{A\chi^2+2B\chi\varrho+C\varrho^2}\Big]\cdot$$

$$\cdot\Big[-\varrho\sqrt{AC-B^2}+\varrho\chi S_{N-1}P_4-\alpha_0'\varrho^2(P_2-\tfrac{1}{2}r_0\alpha_0'P_1)+\tfrac{1}{2}\chi^2 S_{N-1}(S_{N-1}+ \qquad (29)$$

$$+k_*\cos(\varphi_0-\tfrac{2\pi}{N}+\psi_0))\Big]d\chi+\frac{1}{2\pi}\Gamma_m(\varphi_0-\tfrac{2\pi}{N}+\psi_0)\Big[\sqrt{\frac{A}{AC-B^2}}\frac{1}{\varrho}+$$

$$+\frac{1}{2}\frac{1}{\sqrt{A}^3}\left(S_{N-1}^2+S_{N-1}k_*\cos(\varphi_0-\tfrac{2\pi}{N}+\psi_0)\right)\ln\frac{|\varrho|\sqrt{AC-B^2}}{2A\alpha}\Big]\Big\}-$$

$$-\Big[\frac{1}{2\pi}\sqrt{\frac{A}{AC-B^2}}\frac{r-r_0}{k_*^2+(r-r_0)^2}+\frac{1}{4\pi}\frac{1}{\sqrt{A}^3}\left(S_{N-1}^2+S_{N-1}k_*\cos(\varphi_0-\tfrac{2\pi}{N}+\psi_0)\right)\cdot$$

$$\cdot\ln\frac{\sqrt{k_*^2+(r-r_0)^2}\sqrt{AC-B^2}}{2A\alpha}\Big]\Gamma_m(\varphi_0-\tfrac{2\pi}{N}+\psi_0)\cdot$$

Several comments have to be made on this formula (29). The integrand of the first integral is obviously constant. On application of the integral formulas (74) to (81) from No. 7, it further appears that the expression for q = 0 within the braces {} takes the value:

$$+\frac{1}{2\pi}\Gamma_m(\varphi_0-\tfrac{2\pi}{N}+\psi_0)\Big[S_{N-1}P_3\left(\frac{B^2}{\sqrt{AC-B^2}^3}+\frac{1}{2}\frac{1}{\sqrt{AC-B^2}}\right)\frac{1}{\sqrt{A}}-\frac{S_{N-1}P_4B}{(AC-B^2)\sqrt{A}}-$$

$$-\alpha_0'(P_2-\tfrac{1}{2}r_0\alpha_0'P_1)\frac{\sqrt{A}}{AC-B^2}+2\alpha_0'(P_1+\tfrac{r_0}{2}\alpha_0'P_2)\frac{B\sqrt{A}}{\sqrt{AC-B^2}^3}+\frac{r_0\alpha_0'\sqrt{A}^3}{\sqrt{AC-B^2}^3}+$$

$$+\frac{1}{2}\left(S_{N-1}^2+S_{N-1}k_*\cos(\varphi_0-\tfrac{2\pi}{N}+\psi_0)\right)\frac{2B^2-AC}{(AC-B^2)\sqrt{A}^3}+k_*S_{N-1}\sin(\varphi_0-\tfrac{2\pi}{N}+\psi_0)\cdot$$

$$\cdot\left(\frac{BC}{\sqrt{AC-B^2}^3}\frac{1}{\sqrt{A}}+\frac{B}{\sqrt{AC-B^2}^2}\frac{0{,}5}{\sqrt{A}^3}\right)\Big]+\frac{1}{2\pi}\frac{\partial\Gamma_m(\varphi_0-\frac{2\pi}{N}+\psi_0)}{\partial\varphi_0}\frac{B}{\sqrt{AC-B^2}\sqrt{A}}\cdot$$

In the last terms of expression (29) the properly fitting singular expressions $1/Q$, i.e., $\ln|Q|$ have been substituted in $Q/b_*^2+Q^2$, i.e., $\ln\sqrt{b_*^2+Q^2}$ according to the potential theory, by $q = r-r_0$. This procedure is based, of course, on the consideration that an unrolled transverse vortex behaves not as a separate vortex filament in accordance with the potential theory, but rather as a Rankine vortex in a tenacious stream with a finite (even if often very small)

core radius. Such a velocity profile can be quite well approximated with the expression (Figure 9):

$$\varrho\left[\varrho_*^2+\varrho^2\right]^{-1} \qquad (30)$$

with b_* as core radius. Equation (30) is very suitable for further analysis.

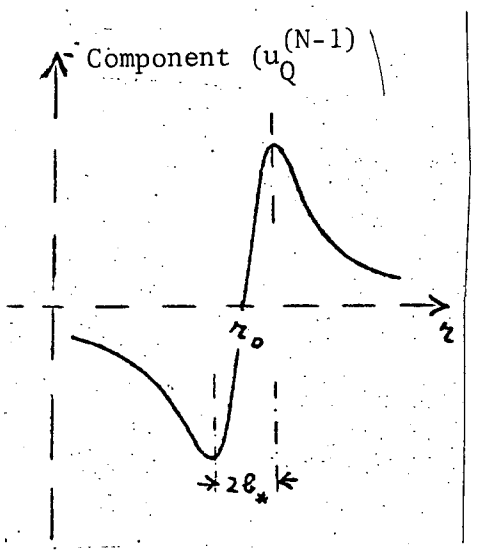

Figure 9. Induced Axial Velocity At The Penetration of a Tip Vortex.

Expression (30) accordingly describes the approximate part of equation (29) essentially as the state of the velocity field when the n = 0 airfoil space point penetrates the unrolled tip vortex and root vortex of the forward-driving blade. /17

If the core radius is small (i.e., $b_* \approx 0.05\ R_o$), the course of the function (30) almost gradually becomes variable. Thus it is to be considered that according to equation (27) both values $r_o \leq R_o$ and also $r_o > R_o$ and the corresponding values $r_i \geq R_i$ and $r_i < R_i$ are possible, depending on the momentary position of the angle ϕ_o of the blade. Since the blade space points r are restricted to the range of $R_i \leq r \leq R_o$, this gradually variable course of function (30) is given only for $r_i \geq R_i$ and for $r_o \leq R_o$. Only then are the unrolled tip vortex and root vortex of the n = N-1 blade of the n = 0 airfoil also actually encountered (Figure 8). These characteristics are to be taken into consideration in connection with the solution theory of the integral equation resulting from the airflow boundary condition.

4. A Few Integral Equations

a) We examine the integral equation:

$$\Gamma_m(\varphi_0 - \tfrac{2\pi}{N} + \psi_0) \cdot f(\varphi_0, \alpha_0) \frac{r - r_0(\varphi_0, \alpha_0)}{\ell_*^2 + [r - r_0(\varphi_0, \alpha_0)]^2} =$$

$$= \frac{1}{2\pi} \int_{R_i}^{R_0} \left[\frac{\partial \Gamma(s, \varphi_0 + \alpha_0)}{\partial s} + \frac{1}{2s} \Gamma(s, \varphi_0 + \alpha_0) \right] \frac{ds}{r - s} \qquad (R_i \leq \frac{r}{s} \leq R_0) \quad (0 \leq \varphi_0 \leq 2\pi) \tag{31}$$

r_0 can also assume values which are larger than R_0. With the transformation:

$$s = \tfrac{1}{2}(R_0 + R_i) - \tfrac{1}{2}(R_0 - R_i)\cos\tau \; ; \quad r = \tfrac{1}{2}(R_0 + R_i) - \tfrac{1}{2}(R_0 - R_i)\cos\vartheta \; ;$$

$$r_0 = \tfrac{1}{2}(R_0 + R_i) - \tfrac{1}{2}(R_0 - R_i)\cos\vartheta_0, \; (r_0 \leq R_0); \quad r_0 = \tfrac{1}{2}(R_0 + R_i) + \tfrac{1}{2}(R_0 - R_i)\operatorname{Col}\vartheta_0^*, \; (r_0 \geq R_0)$$

equation (31) passes into the form:

$$-2\Gamma_m(\varphi_0 - \tfrac{2\pi}{N} + \psi_0) f(\varphi_0, \alpha_0) \frac{\cos\vartheta + \operatorname{Col}\vartheta_0^*}{\ell^2 + (\cos\vartheta + \operatorname{Col}\vartheta_0^*)^2} =$$

$$= \frac{1}{\pi} \int_0^{\pi} \left(\frac{\partial \Gamma(\tau, \varphi_0 + \alpha_0)}{\partial \tau} + \frac{1}{2} \frac{\sin\tau \cdot \Gamma(\tau, \varphi_0 + \alpha_0)}{\frac{R_0 + R_i}{R_0 - R_i} - \cos\tau} \right) \frac{d\tau}{\cos\tau - \cos\vartheta} \tag{32}$$

$$(0 \leq \frac{\vartheta}{\tau} \leq \pi \; ; \quad \vartheta_0 \leq \pi \; ; \quad \vartheta_0^* \geq 0) \; .$$

In equation (32) $-\cos\vartheta_0$ is equivalent to $r_0 \leq R_0$ and τ or ϑ_0^* to $r_0 \geq R_0$. /18
At the transition point $\vartheta_0 = \pi$ and $\vartheta_0^* = 0$ both formulas are in agreement, as is required.

With the integral formula demonstrated in No. 7 there follows next for the solution of the integral equation (32):

$$\frac{\partial \Gamma(\tau, \varphi_0 + \alpha_0)}{\partial \tau} + \frac{1}{2} \frac{\sin\tau \cdot \Gamma(\tau, \varphi_0 + \alpha_0)}{\frac{R_0 + R_i}{R_0 - R_i} - \cos\tau} = \ell\sqrt{2}\, \Gamma_m(\varphi_0 - \tfrac{2\pi}{N} + \psi_0) f(\varphi_0, \alpha) \cdot$$

$$\cdot \frac{\sqrt{(\Xi^2 - 1 - \ell^2)^2 + 4\Xi^2\ell^2} + \sin^2\tau}{\ell^2 + (\cos\tau - \Xi)^2} \left[1 - \Xi^2 + \ell^2 + \sqrt{(\Xi^2 - 1 - \ell^2)^2 + 4\Xi^2\ell^2} \right]^{-\frac{1}{2}} +$$

$$+ \ell\, S(\varphi_0, \alpha_0) \; ; \qquad (S \text{ initially undetermined}) \; . \tag{33}$$

$$\Xi = \cos\vartheta_0 \text{ für } r_0 \leq R_0 \; ; \qquad \Xi = -\operatorname{Col}\vartheta_0^* \text{ for } r_0 \geq R_0$$

The values ϑ_o and ϑ_o^* dependent like r_o on the momentary angle position ϕ_o. Equation (33) is to be regarded, with reference to its τ-dependence, as the usual linear differential equation for the sought function Γ. It can be integrated directly with the known solution formula. The unknown function ζ (with respect to the variable τ, ζ is a constant), as well as the integration constant appearing further with the integration of such as equation (33), are to be determined from the condition (20) necessary to be fulfilled, consequently $\Gamma = 0$ for $\tau = 0$ and $\tau = \pi$.

By this means, after a simple intermediate computation, the sought solution of integral equation (31), i.e., (32), takes the following form:

$$\Gamma(\tau, \varphi_o + \alpha_o) = \frac{b\,\Gamma_m(\varphi_o - \frac{2\pi}{N} + \psi_o)\, f(\varphi_o, \alpha_o)\left[\frac{R_o+R_i}{R_o-R_i} - \cos\tau\right]^{-1/2}}{\left[1 - \Xi^2 + b^2 + \sqrt{(\Xi^2 - 1 - b^2)^2 + 4\Xi^2 b^2}\right]^{1/2}} \cdot$$

$$\cdot \left\{ \int_0^\tau \frac{\left[\frac{R_o+R_i}{R_o-R_i} - \cos\tau'\right]^{1/2}}{b^2 + (\cos\tau' - \Xi)^2} \sqrt{2}\left(\sqrt{(\Xi^2 - 1 - b^2)^2 + 4\Xi^2 b^2} + \sin^2\tau'\right) d\tau' - \right.$$

$$- \int_0^\tau \left[\frac{R_o+R_i}{R_o-R_i} - \cos\tau'\right]^{1/2} d\tau \cdot \left(\int_0^\pi \left[\frac{R_o+R_i}{R_o-R_i} - \cos\tau'\right]^{1/2} d\tau' \right)^{-1} \cdot \qquad (34)$$

$$\left. \cdot \int_0^\pi \frac{\left[\frac{R_o+R_i}{R_o-R_i} - \cos\tau'\right]^{1/2}}{b^2 + (\cos\tau' - \Xi)^2} \sqrt{2}\left(\sqrt{(\Xi^2 - 1 - b^2)^2 + 4\Xi^2 b^2} + \sin^2\tau'\right) d\tau' \right\} .$$

with $\Xi = \cos\vartheta_o$ für $r_o \leq R_o$, $\Xi = -\cos\vartheta_o^*$ für $r_o \geq R_o$.

Abbreviation: $\Gamma(\tau, \varphi_o + \alpha_o) = \Gamma_m(\varphi_o - \frac{2\pi}{N} + \psi_o)\, f(\varphi_o, \alpha_o)\, \gamma(\tau, \varphi_o)$.

$f(\phi_o, \alpha_o)$ is an optional, continuous function. /19

Quite analogously, the corresponding integral equation (31) can be solved when $r_i(\phi_o, \alpha_o)$ appear in place of r_o. There is then $\Xi = \cos \vartheta_o$ for $r_i \geq R_i$? and $\Xi = +$ of ϑ_o with $\vartheta_o^* \geq 0$ for $r_i \leq R_i$. ??

It appears expedient to follow this up with a view of the order of magnitude obtained in solution (34).

b_* is to be considered the core radius of an unrolled transverse vortex by further application of formula (34).

From the results of measurements it is known [4] that the core radius of an unrolled tip vortex of a rotor lies perhaps between 0.04 R_o and 0.07 R_o. Hence it follows that with $b_* = 1/2\,(R_o - R_i)b$

$$b^2 \approx 10^{-2}\,.$$

Since ordinarily $(R_o - R_i)^2/(R_o + R_i)^2 \lesssim 1/2$ will be present, then

$$\int_0^\tau \sqrt{\frac{R_o+R_i}{R_o-R_i} - \cos\tau'}\,d\tau' \approx \sqrt{\frac{R_o+R_i}{R_o-R_i}}\left(\tau - \frac{1}{2}\,\frac{R_o-R_i}{R_o+R_i}\sin\tau\right). \qquad (35)$$

is valid to a good approximation.

By estimating the order of magnitude of solution (34) for small values of b, we will distinguish various cases. To begin with, there is $\sin^2\vartheta_o \gg b^2$. Then with $\Xi = \cos\vartheta_o$, it becomes

$$\frac{\Gamma(\tau,\varphi_o+\alpha_o)}{\Gamma_m \cdot f(\varphi_o,\alpha_o)} \approx b\left(\frac{R_o+R_i}{R_o-R_i} - \cos\tau\right)^{-1/2}\left\{\int_0^\tau\left(\frac{R_o+R_i}{R_o-R_i} - \cos\tau'\right)^{1/2}\cdot\right.$$
$$\cdot\frac{\left(\sin\vartheta_o + \frac{\sin^2\tau'}{\sin\vartheta_o}\right)d\tau'}{b^2+(\cos\tau'-\cos\vartheta_o)^2} - \int_0^\tau\left(\frac{R_o+R_i}{R_o-R_i} - \cos\tau'\right)^{1/2}d\tau'\Big/\int_0^\pi\left(\frac{R_o+R_i}{R_o-R_i} - \cos\tau'\right)d\tau'\Big)^{-1}\cdot \qquad (36)$$
$$\left.\cdot\int_0^\pi\left(\frac{R_o+R_i}{R_o-R_i} - \cos\tau'\right)^{1/2}\frac{\left(\sin\vartheta_o + \frac{\sin^2\tau'}{\sin\vartheta_o}\right)d\tau'}{b^2+(\cos\tau'-\cos\vartheta_o)^2}\right\}.$$

The definite integrand in formula (36) obviously has a steep maximum when $\tau' \approx \vartheta_o$, and is in good agreement ($\varepsilon^2 \ll 1$)

$$\int_0^\pi\left(\frac{R_o+R_i}{R_o-R_i} - \cos\tau'\right)^{1/2}\frac{\left(\sin\vartheta_o + \frac{\sin^2\tau'}{\sin\vartheta_o}\right)d\tau'}{b^2+(\cos\tau'-\cos\vartheta_o)^2} \approx \int_{\vartheta_o-\varepsilon}^{\vartheta_o+\varepsilon}(\ldots)\,d\tau' \approx \qquad (37)$$
$$\approx 2\sin\vartheta_o\left(\frac{R_o+R_i}{R_o-R_i} - \cos\vartheta_o\right)^{1/2}\int_{\vartheta_o-\varepsilon}^{\vartheta_o+\varepsilon}\frac{d\tau'}{b^2+\sin^2\vartheta_o\cdot(\tau'-\vartheta_o)^2} = \frac{4}{b}\left(\frac{R_o+R_i}{R_o-R_i} - \cos\vartheta_o\right)^{1/2}\operatorname{arc\,tan}\frac{\varepsilon\sin\vartheta_o}{b}$$

With (35) and (37), the estimation from equation (36) then follows: /20

$$\gamma = \frac{\Gamma}{\Gamma_m \cdot f} \approx \frac{\left(\frac{R_o+R_i}{R_o-R_i} - \cos\vartheta_o\right)^{1/2}}{\left(\frac{R_o+R_i}{R_o-R_i} - \cos\tau\right)^{1/2}} \, 4 \arctan\left|\frac{\varepsilon \sin\vartheta_o}{b}\right| \cdot \tag{39}$$

$$\cdot \begin{cases} 1 - \frac{1}{\pi}\left(\tau - \frac{1}{2}\frac{R_o-R_i}{R_o+R_i}\sin\tau\right) & \text{for } \tau > \vartheta_o \\ -\frac{1}{\pi}\left(\tau - \frac{1}{2}\frac{R_o-R_i}{R_o+R_i}\sin\tau\right) & \text{for } \tau < \vartheta_o \end{cases} \quad (\sin^2\vartheta_o \gg b^2)$$

Formula (38) shows that even when $b \to 0$, $\Gamma/\Gamma_m, f \sim 0$ (1) remains. Besides, $\Gamma/\Gamma_m, f$ is positive for $r > r_o$, negative for $r < r_o$ (as to be expected according to the structure of the integral equation (31)). This is obviously clear if one thinks of the induced velocity from an unrolled tip vortex. ($\Gamma/\Gamma_m, f = \gamma$). (Figure 10). Now let $\vartheta_o = \pi$ or $\vartheta_o^* = 0$. Then with $\Xi = -1$,

$$\frac{\Gamma(\tau, \varphi_o+\alpha_o)}{\Gamma_m \cdot f(\varphi_o, \alpha_o)} \approx \frac{b}{\sqrt{b+\frac{1}{2}b^2}} \left(\frac{R_o+R_i}{R_o-R_i} - \cos\tau\right)^{-1/2} \left\{ \int_o^\tau \left(\frac{R_o+R_i}{R_o-R_i} - \cos\tau'\right)^{1/2} \cdot \right.$$

$$\cdot \frac{(2b + \sin^2\tau')\,d\tau'}{b^2 + (\cos\tau'+1)^2} - \int_o^\tau \left(\frac{R_o+R_i}{R_o-R_i} - \cos\tau'\right)^{1/2} d\tau' \left(\int_o^\pi \left(\frac{R_o+R_i}{R_o-R_i} - \cos\tau'\right)^{1/2} d\tau' \right)^{-1} \cdot \tag{39}$$

$$\left. \cdot \int_o^\pi \left(\frac{R_o+R_i}{R_o-R_i} - \cos\tau'\right)^{1/2} \frac{(\sin^2\tau' + 2b)\,d\tau'}{b^2 + (\cos\tau'+1)^2} \right\} .$$

Considering the relationship (see formula (65), No. 7)

$$\frac{1}{\pi} \int_o^\pi \left(1 - \frac{1}{2}\frac{R_o-R_i}{R_o+R_i}\cos\tau\right) \frac{(\sin^2\tau + 2b)\,d\tau}{b^2 + (\cos\tau+1)^2} \approx \frac{1}{\sqrt{b}}\left(2 + \frac{R_o-R_i}{R_o+R_i}\right) - 1 - \frac{R_o-R_i}{R_o+R_i} \tag{40}$$

as well as equation (35), we obtain from (39) the evaluation:

$$\frac{\Gamma}{\Gamma_m \cdot f} \approx \sqrt{b}\left(\frac{R_o+R_i}{R_o-R_i} - \cos\tau\right)^{-1/2} \left\{ \int_o^\tau \left(\frac{R_o+R_i}{R_o-R_i} - \cos\tau'\right)^{1/2} \frac{(2b + \sin^2\tau')\,d\tau'}{b^2 + (\cos\tau'+1)^2} - \right.$$

$$\left. - \left(\tau - \frac{1}{2}\frac{R_o-R_i}{R_o+R_i}\sin\tau\right) \frac{1}{\sqrt{b}} \left(2 + \frac{R_o-R_i}{R_o+R_i} - \sqrt{b} - \sqrt{b}\,\frac{R_o-R_i}{R_o+R_i}\right) \right\} \approx \gamma ; \tag{40}$$

$$\text{for } \tau < \vartheta_o \,, \quad \vartheta_o = \pi, \quad \vartheta_o^* = 0 \,.$$

Also from expression (40) there follows: $\Gamma/\Gamma_m, f \quad 0(1)$ for $b \to 0$. Understandably, $\Gamma/\Gamma_m f$ is negative, since in (40) $r_o = R_o$, then $r < r_o$. We finally consider the case in which $\gamma_m^2 \vartheta_o^* >> b^2$ and $\gamma_m \vartheta_o^* \simeq |\text{cotan } \vartheta_o^*$, i.e., the unrolled transverse vortex lies at some distance beyond the airfoil (Figure 10). For $b \to 0$, there is obtained from (34): /21

$$\gamma = \frac{\Gamma}{\Gamma_m \cdot f} \approx \left(\frac{R_o+R_i}{R_o-R_i} - \cos\tau\right)^{-1/2} \left\{ \int_0^{\tau} \left(\frac{R_o+R_i}{R_o-R_i} - \cos\tau'\right)^{1/2} \frac{(\gamma_m^2 \vartheta_o^* + \sin^2\tau')\,d\tau'}{(\text{cotan}\,\vartheta_o^* + \cos\tau')^2} - \right.$$

$$\left. - \frac{1}{\pi}\left(\tau - \frac{1}{2}\frac{R_o-R_i}{R_o+R_i}\sin\tau\right) \int_0^{\pi} \left(\frac{R_o+R_i}{R_o-R_i} - \cos\tau'\right)^{1/2} \frac{(\gamma_m^2 \vartheta_o^* + \sin^2\tau')\,d\tau'}{(\text{cotan}\,\vartheta_o^* + \cos\tau')^2} \right\}. \qquad (41)$$

With increasing ϑ_o^* (hence with growing distance of the unrolled transverse vortex from the airfoil) $\Gamma/\Gamma_m, f \to 0$ follows from (41), as it must. Then $(\gamma_m^2 + \vartheta_o^2\ \tau')(\text{Cotan}\ \vartheta_o^* + \cos\tau')^{-2}$ becomes independent of τ', and therefore the integrals over τ' can be approximated by means of (35). With this, the discussion of the rigorous solution (34) from the integral equation (31) is concluded. Quite analogously, the solution of the corresponding integral equation (31) permits investigation with $r_i(\phi_o, \alpha_o)$ in place of $r_o(\phi_o, \alpha_o)$.

The derivative $\partial\Gamma/\partial\phi_o$ of solution (34) remains constant at the transition point $r_o = R_o$, or else at $\vartheta_o = \pi$ and $\vartheta_o^* = 0$ according to the angle ϕ_o, which amounts to a parameter. Then once $dr_o/d\phi_o$ is constant, in conformance with equation (27), the reduction of terms in (34) for the second time is certainly valid.

$$r_o = \frac{1}{2}(R_o+R_i) - \frac{1}{2}(R_o-R_i)\,\Xi\ ,$$

and from this it follows that

$$\frac{d\Xi}{d\varphi_o} = -\frac{2}{R_o-R_i}\frac{dr_o}{d\varphi_o}\ ;$$

and consequently $\partial\Gamma/\partial\phi_o$ remains constant in the area of $0 \leq \phi_o \leq 2\pi$.

b) We examine the integral equation ($R_i \leq r/s \leq R_o$; $0 \leq \phi_o \leq 2\pi$):

$$F(r,\varphi_o,\alpha) + \Gamma_m(\varphi_o - \tfrac{2\pi}{N} + \psi_o)\, f(\varphi_o,\alpha_o)\, \frac{r - r_o(\varphi_o,\alpha_o)}{\varrho_*^2 + [r - r_o(\varphi_o,\alpha_o)]^2} =$$

$$= \frac{1}{2\pi}\int_{R_i}^{R_o}\left[\frac{\partial\Gamma(s,\varphi_o+\alpha_o)}{\partial s} + \frac{1}{2s}\Gamma(s,\varphi_o+\alpha_o)\right]\frac{ds}{r-s} +$$

$$+ \frac{1}{2\pi} f_*(r,\varphi_o)\int_{R_i}^{R_o}\frac{\partial\Gamma(s,\varphi_o - \frac{2\pi}{N} + \psi_{N-1})}{\partial\varphi_o}\,\frac{ds}{s - s^*_{N-1}(r,\varphi_o)} + q(r,\varphi_o)\,\Gamma_m(\varphi_o - \tfrac{2\pi}{N} + \psi_o) \qquad (42)$$

$$+ \frac{1}{2\pi}\int_{\chi_A}^{\chi_E}\Gamma_m(\varphi_o - \tfrac{2\pi}{N} + \psi_o + \chi)\, Q(r,\varphi_o,\chi)\,d\chi + \frac{1}{2\pi}\int_{R_i}^{R_o}\Big\{\Gamma(s,\varphi_o+\Theta_1)K_1(r,s,\varphi_o) +$$

$$+ \frac{\partial\Gamma(s,\varphi_o+\Theta_2)}{\partial s}K_2(r,s,\varphi_o) + \frac{\partial\Gamma(s,\varphi_o+\Theta_3)}{\partial\varphi_o}K_3(r,s,\varphi_o) +$$

$$+ \int_{(\chi)}\left[\frac{\partial\Gamma(s,\varphi_o+\Theta_4+\chi)}{\partial s}K_4(r,s,\varphi_o,\chi) + \frac{\partial\Gamma(s,\varphi_o+\Theta_5+\chi)}{\partial\varphi_o}K_5(r,s,\varphi_o,\chi)\right]d\chi\Big\}\,ds.$$

The expansion of equation (42) still requires a few comments. Equation (42) mainly contains the integral equation (31) as a component. There occurs, moreover, a special singular core portion of the novel from $[S - S^*_{N-1}(r,\phi_o)]^{-1}$. Therefore $S^*_{N-1}(r, \phi_o)$ is given by equation (23). The values $R_i \leq S^*_{N-1} \leq R_o$ and also $S^*_{N-1} < R_i$, $S^*_{N-1} > R_o$ are possible. For the latter the core portion mentioned is not singular. /22

Γ_m is a special value of the sought function Γ. In fact, it is a question of the circulation of the unrolled tip vortex and root vortex with the application of the rotor airflow. However, we come back to the proper determination of Γ_m.

K_1 to K_5 as well as Q are constant core portions in all the variables (42). $\int_{(\chi)}$ is a written symbolic expression for integrals over χ with various possible limits, whose existence is thereby naturally assumed. Θ_1 to Θ_5 denote phase displacements of the ϕ_o-slope. $q(r, \phi_o)$, $f_*(r, \phi_o)$, $f(\phi_o\ \alpha_o)$, $F(r, \phi_o, \alpha_o)$ represent suggested continuous functions.

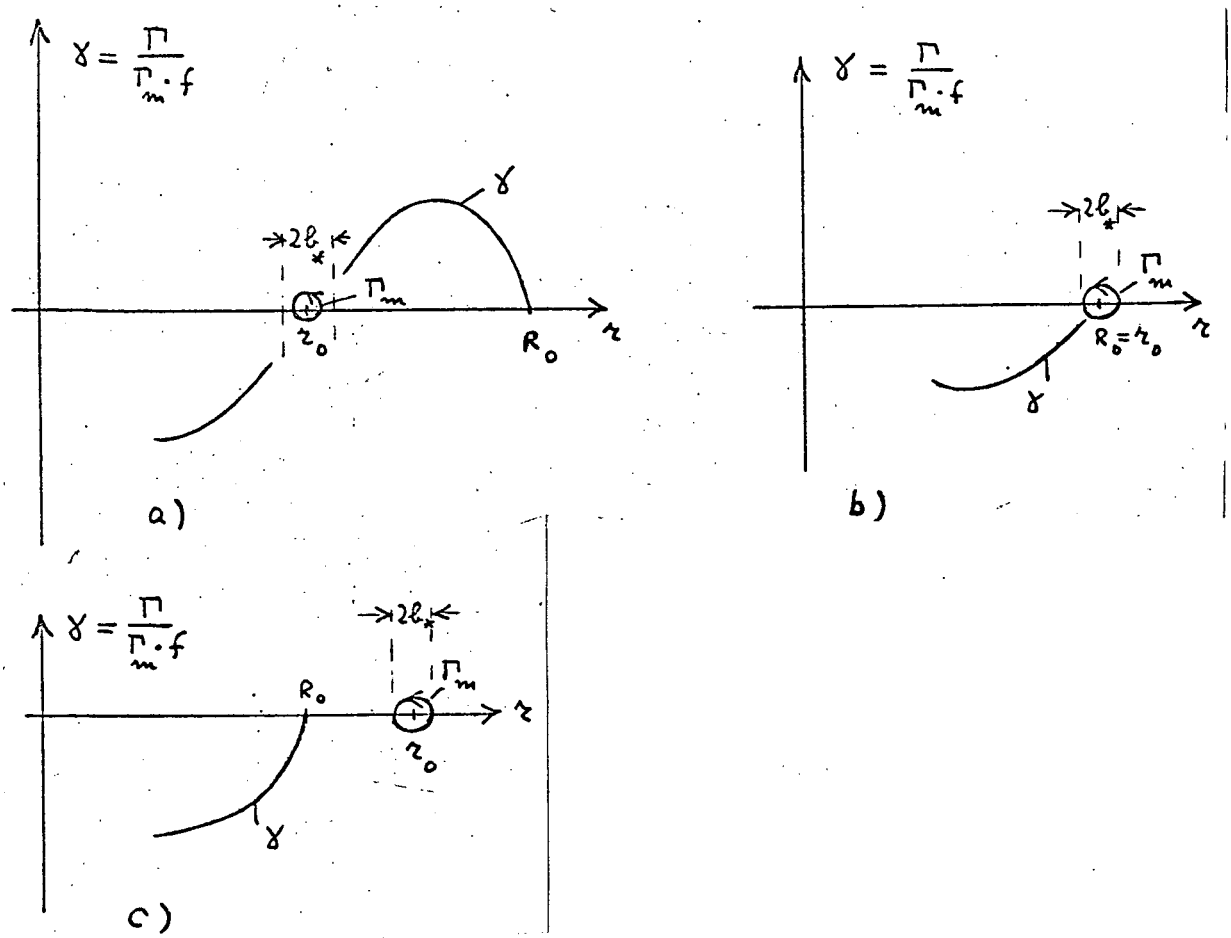

Figure 10. Various Positions of Tip Vortices From Blade N-1 At The Airfoil Space Point. Generated Circulation Portion $\gamma = \Gamma/\Gamma_m, f$ From Tip Vortex at Blade 0.

For the solution of equation (42) we first produce with the transformation

$$s = \frac{1}{2}(R_0+R_i) - \frac{1}{2}(R_0-R_i)\cos\tau \quad , \quad (\varphi_\alpha = \varphi_0 - \alpha_0)$$

the statement

$$\Gamma(s,\varphi_0) \equiv \Gamma(\tau,\varphi_0) = \Gamma_m(\varphi_\alpha - \tfrac{2\pi}{N} + \psi_0) f(\varphi_\alpha,\alpha_0)\,\gamma(\tau,\varphi_\alpha) + \Gamma_*(\tau,\varphi_0). \qquad (43)$$

That is, let $\Gamma_m\,(\phi_0 - 2\pi/N + \psi_0) f(\phi_0, \alpha_0)\gamma(\tau, \phi_0)$ be the solution of the integral equation (31). The function γ is therefore known, and can be directly

taken from formula (34). In the case of the rotor air stream, γ characterizes the influence of a penetrating unrolled transverse vortex from the rotor blade on the airfoil circulation. Since $\gamma(o, \phi_o) = 0$, so is $\gamma(\pi, \phi_o) = 0$, then γ can be developed in a Fourier sine series (also by a relatively steep course of the function in the presence of a corresponding L number of terms):

$$\gamma(\tau, \varphi_o) = \sum_{\lambda=1}^{L} \gamma_\lambda(\varphi_o) \sin \lambda\tau \quad . \tag{44}$$

The additional function development Γ_* is determined essentially by the remaining velocity field in which the airfoil blade operates. Also, Γ_* must disappear for $\tau = 0$ and $\tau = \pi$, and should be put into a Fourier series in the form:

$$\Gamma_*(\tau, \varphi_o) = \sum_{\mu=-M_o}^{M_o} \sum_{\nu=1}^{M} a_{\mu\nu} e^{i\mu\varphi_o} \sin \nu\tau \quad .$$

$$(0 \leq \tau \leq \pi \; , \quad 0 \leq \varphi_o \leq 2\pi) \quad . \tag{45}$$

We would like to begin with the idea that the unrolling process of the free vortex perhaps results in such a way that the existing vortex substance in the area of $r > R_o + R_i/2$ builds up the tip vortex, and that in the area of $r < R_o + R_i/2$ it builds up the root vortex. Then[4]: /23

$$\Gamma_m(\varphi_o) \approx \Gamma\left(\frac{R_o + R_i}{2}, \varphi_o\right) = \Gamma\left(\frac{\pi}{2}, \varphi_o\right)$$

Setting $\psi_o \approx 2\pi/N$ as an approximation, then there follows from (43) (neglecting α_o):

$$\Gamma_m(\varphi_o) \approx \Gamma_m(\varphi_o) f(\varphi_o, \alpha_o) \gamma\left(\frac{\pi}{2}, \varphi_o\right) + \Gamma_*\left(\frac{\pi}{2}, \varphi_o\right) \; ,$$

or

$$\Gamma_m(\varphi_o) = \frac{\Gamma_*\left(\frac{\pi}{2}, \varphi_o\right)}{1 - f(\varphi_o, \alpha_o) \gamma\left(\frac{\pi}{2}, \varphi_o\right)} \quad . \tag{46}$$

[4]Fundamentally, a τ-value other than $\pi/2$ can also naturally be used for the determination of Γ_m.

Since after the calculation of (38) $\gamma(\pi/2, \phi_0) < 0$ results, and $f > 0$ can also be assumed according to equation (29), so that the denominator in (46) does not disappear.

Before we go into the integral equation (42) with the statements (43) to (46), we would like to collate a few more formulas for rational computation of the singular core portion in equation (42). First, this transformation is carried out:

$$
\begin{aligned}
&s^*_{N-1} = \frac{R_0+R_i}{2} - \frac{R_0-R_i}{2}\cos\xi \quad , \quad \text{for} \quad R_i \le s^*_{N-1} \le R_0 \quad ; \\
&s^*_{N-1} = \frac{R_0+R_i}{2} \pm \frac{R_0-R_i}{2}\cos\xi^* \quad ; \qquad + \text{ für } s^*_{N-1} \ge R_0 \qquad (\xi^* \ge 0) \\
&r = \frac{R_0+R_i}{2} - \frac{R_0-R_i}{2}\cos\vartheta \quad . \qquad\qquad - \text{ für } s^*_{N-1} \le R_i \quad ;
\end{aligned}
\tag{47}
$$

Consideration of integral formulas (66) to (68) from No. 7 then leads to:

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{R_i}^{R_0}\left\{\frac{\partial\Gamma_*(s,\varphi_0+\alpha_0)}{\partial s} + \frac{1}{2s}\Gamma_*(s,\varphi_0+\alpha_0)\right\}\frac{ds}{r-s} = \frac{1}{2\pi}\int_{R_i}^{R_0}\Gamma_*(s,\varphi_0+\alpha)\frac{ds}{2rs} + \\
&+ \frac{1}{2\pi}\int_{R_i}^{R_0}\left\{\frac{\partial\Gamma_*(s,\varphi_0+\alpha_0)}{\partial s} + \frac{1}{2r}\Gamma_*(s,\varphi_0+\alpha_0)\right\}\frac{ds}{r-s} = \\
&= \sum_{\mu=-M_0}^{M_0}\sum_{\nu=1}^{R_i M} a_{\mu\nu}\, e^{i\mu(\varphi_0+\alpha_0)}\left\{\frac{1}{R_0-R_i}\frac{\sin\nu\vartheta}{\sin\vartheta} - \frac{1}{4\nu r(\vartheta)}\cos\nu\vartheta + \right. \\
&\qquad \left. + \frac{1}{2\pi\nu}\int_0^{\pi}\frac{R_0-R_i}{4s(\tau)r(\vartheta)}\sin\nu\tau\sin\tau\,d\tau\right\} \quad . \qquad (R_i \le \tfrac{r}{s} \le R_0) \quad .
\end{aligned}
\tag{48}
$$

$$
\frac{1}{2\pi}\int_{R_i}^{R_0}\frac{\partial\Gamma(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1})}{\partial\varphi_0}\,\frac{ds}{s-s^*_{N-1}(r,\varphi_0)} = \tag{49}
$$

$$= \frac{1}{2} \sum_{\mu=-M_o}^{M_o} \sum_{\nu=1}^{M} a_{\mu\nu} \cdot i\mu \, e^{i\mu(\varphi_o - \frac{2\pi}{N} + \Psi_{N-1})} \begin{pmatrix} \cos\nu\xi \\ e^{-\nu\xi^*} \\ (-1)^\nu e^{-\nu\xi^*} \end{pmatrix} +$$

$$+ \frac{1}{2} \sum_{\mu=-M_o}^{M_o} \sum_{\nu=1}^{M} a_{\mu\nu} \sin\nu\tfrac{\pi}{2} \cdot \frac{\partial}{\partial\varphi_o} \left\{ e^{i\mu(\varphi_\alpha - \frac{4\pi}{N} + \Psi_{N-1} + \Psi_o)} \cdot f(\varphi_\alpha - \tfrac{2\pi}{N} + \Psi_{N-1}, \alpha_o) \cdot \right.$$

$$\left. \cdot \sum_{\lambda=1}^{L} \gamma_\lambda(\varphi_\alpha - \tfrac{2\pi}{N} + \Psi_{N-1}) \left[1 - f(\varphi_\alpha - \tfrac{4\pi}{N} + \Psi_{N-1} + \Psi_o) \cdot \gamma(\tfrac{\pi}{2}, \varphi_\alpha - \tfrac{4\pi}{N} + \Psi_{N-1} + \Psi_o) \right]^{-1} \right\} \cdot$$

$$\cdot \begin{pmatrix} \cos\lambda\xi \\ e^{-\lambda\xi^*} \\ (-1)^\lambda e^{-\lambda\xi^*} \end{pmatrix} \qquad (\varphi_\alpha = \varphi_o - \alpha_o). \tag{49}$$

In formula (49) the upper line in the expression between parentheses holds for $R_i \leq S^*_{N-1} \leq R_o$, the middle line for $S^*_{N-1} \leq R_i$, and the bottom line for $S^*_{N-1} \geq R_o$[5]. In the core of the rotor airflow, for $0 \leq \phi_o + \alpha \leq \pi$ both $R_i \leq S^*_{N-1} \leq R_o$ and also $S^*_{N-1} \leq R_i$, and for $\pi \leq \phi_o + \alpha \leq 2$, both $R_i < S^*_{N-1} \leq R_o$ and also $S^*_{N-1} \geq R_o$ are possible. (See equation (23)). The variables ξ, ξ^* transformed according to formula (47) then lie in the following regions:

$$0 \leq \varphi_o + \alpha \leq \pi : \quad 0 \leq \xi \leq \xi_1(\varphi_o) \quad \text{and} \quad 0 \leq \xi^* \leq \xi_1^*(\varphi_o) ;$$

$$\pi \leq \varphi_o + \alpha \leq 2\pi : \quad \xi_2(\varphi_o) \leq \xi \leq \pi \quad \text{and} \quad 0 \leq \xi^* \leq \xi_2^*(\varphi_o) ;$$

with:

$$\xi_1(\varphi_o) = \arccos\left(\frac{R_o + R_i - 2S^*_{N-1}(R_o, \varphi_o)}{R_o - R_i} \right) ; \quad \xi_1(-\alpha) = \xi_1(\pi - \alpha) = \pi.$$

$$\xi_1^*(\varphi_o) = \operatorname{Arcosh}\left(\frac{R_o + R_i - 2S^*_{N-1}(R_i, \varphi_o)}{R_o - R_i} \right) ; \quad \xi_1^*(-\alpha) = \xi_1^*(\pi - \alpha) = 0.$$

$$\xi_2(\varphi_o) = \arccos\left(\frac{R_o + R_i - 2S^*_{N-1}(R_i, \varphi_o)}{R_o - R_i} \right) ; \quad \xi_2(-\alpha) = \xi_2(\pi - \alpha) = 0.$$

$$\xi_2^*(\varphi_o) = \operatorname{Arcosh}\left(\frac{R_o + R_i - 2S^*_{N-1}(R_o, \varphi_o)}{R_i - R_o} \right) ; \quad \xi_2^*(-\alpha) = \xi_2^*(\pi - \alpha) = 0. \tag{50}$$

[5]The values in the expression enclosed in parentheses constantly merge at the transition points $\xi = 0$, $\xi^* = 0$ or $\xi = \pi$, $\xi^* = 0$. $S^*_{N-1}(r, \phi)$. The derivation $\partial/\partial\phi_o$ in (49) is to be carried out only after the explicitly occurring ϕ_o-values, but nor after the also ϕ_o-dependent phase displacements Ψ_{N-1} and ψ_o. (Then equation (46) would thus also be obtained independent of ϕ_o under the assumption of $\psi_o \approx 2\pi/N$).

The solution formulations (43) to (46) for the integral equation (42) can be represented in the more distinct, linear dependent form by the solution coefficients Q_{MV} to be calculated.

$$\Gamma(s,\varphi_0) \equiv \Gamma(\tau,\varphi_0) = \sum_{\mu=-M_0}^{M_0} \sum_{\nu=1}^{M} a_{\mu\nu}\, e^{i\mu\varphi_0} \left\{ \sin\nu\tau + \right.$$

$$\left. + \frac{e^{-i\mu\left(\frac{2\pi}{N}-\psi_0+\alpha_0\right)} \sin\nu\frac{\pi}{2} \cdot f(\varphi_d,\alpha_0)\cdot\gamma(\tau,\varphi_d)}{1 - f\left(\varphi_\alpha - \frac{2\pi}{N}+\psi_0,\alpha_0\right)\gamma\left(\frac{\pi}{2},\varphi_\alpha-\frac{2\pi}{N}+\psi_0\right)} \right\} . \qquad [a_{-\mu,\nu} = \bar{a}_{\mu\nu}] . \quad (51)$$

/25

By this $\gamma(\tau, \phi_0)$ describes the previously known solution (34) of the integral equation (31). It characterizes in connection with the rotor flow-through the effect of the penetrating unrolled transverse vortex (tip vortex) of the forward-driving blades on the circulation distribution at the airfoil. The formulation in the form (51) shows this part of the separated solution especially clear. The rest of the solution consists of a conventional theoretical line-support sin $\nu\tau$-series. With (51) the integral equation (42) changes to the form:

Integral Equation

$$F(r,\varphi_0,\alpha) \equiv \Gamma(r,\varphi_0) - \frac{1}{2\pi}\int_{R_i}^{R_0}\left[\frac{\partial\Gamma_*(s,\varphi_0+\alpha_0)}{\partial s} + \frac{1}{2s}\Gamma_*(s,\varphi_0+\alpha_0)\right]\frac{ds}{r-s} +$$

$$+ \frac{1}{2\pi} f_*(r,\varphi_0)\int_{R_i}^{R_0}\frac{\partial\Gamma(s,\varphi_0-\frac{2\pi}{N}+\Psi_{N-1})}{\partial\varphi_0}\frac{ds}{s^*_{N-1}(r,\varphi_0)-s} -$$

$$- \frac{q(r,\varphi_0)\,\Gamma_*\left(\frac{\pi}{2},\varphi_0-\frac{2\pi}{N}+\psi_0\right)}{1-f\left(\varphi_0-\frac{2\pi}{N}+\psi_0,\alpha_0\right)\gamma\left(\frac{\pi}{2},\varphi_0-\frac{2\pi}{N}+\psi_0\right)} - \frac{1}{2\pi}\int_{\chi_A}^{\chi_E} Q(r,\varphi_0,\chi)\cdot \quad (52)$$

$$\cdot \frac{\Gamma_*\left(\frac{\pi}{2},\varphi_0-\frac{2\pi}{N}+\psi_0+\chi\right)d\chi}{1-f\left(\varphi_0-\frac{2\pi}{N}+\psi_0+\chi,\alpha_0\right)\gamma\left(\frac{\pi}{2},\varphi_0-\frac{2\pi}{N}+\psi_0+\chi\right)} - \frac{1}{2\pi}\int_{R_i}^{R_0}\left\{\Gamma(s,\varphi_0+\Theta_1)K_1(r,s,\varphi_0)+\right.$$

$$+ \frac{\partial\Gamma(s,\varphi_0+\Theta_2)}{\partial s}K_2(r,s,\varphi_0) + \frac{\partial\Gamma(s,\varphi_0+\Theta_3)}{\partial\varphi_0}K_3(r,s,\varphi_0) +$$

$$\left. + \int_{(\chi)}\left[\frac{\partial\Gamma(s,\varphi_0+\Theta_4+\chi)}{\partial s}K_4(r,s,\varphi_0,\chi) + \frac{\partial\Gamma(s,\varphi_0+\Theta_5+\chi)}{\partial\varphi_0}K_5(r,s,\varphi_0,\chi)\right]d\chi\right\}ds =$$

$$= 0 . \qquad (0\le\varphi_0\le 2\pi\,;\ R_i\le r\le R_0) .$$

It appears most practical to carry out the solution of (52) according to the method of least squares, with this restriction therefore:

$$\sum_{\ell=1}^{2L_o} \sum_{j=1}^{L} \left\{ Ig1(r_j, \varphi_{o\ell}) \right\}^2 = Minimum . \qquad (53)$$

Thereby r_j, ϕ_{o1} are chosen space points. As a rule, $\phi_{o1} = 1\pi/L_o$ and $r_j = 1/2(R_o + R_i) - 1/2(R_o - R_i)\cos j\pi/L + 1$.

With the formulation in (51) the condition (53) leads to a linear equation system for calculating the $2M_o \cdot M$ solution coefficients a_{MV}. In the special case where $L + M$ and $L_o = M_o$, the minimum disappears. Usually, however, $L > M$, and under the circumstances also $L_o > M_o$, will be chosen in order to reach a most likely equivalent approximation. /26

Both singular integrands in formula (52) are evaluated with the aid of formulas (48) and (49), the remaining completely fixed integrands, using one of the usual quadrature treatments. The fact that, according to the r_j, ϕ_{o1} space point present, either $R_i \leq S^*_{N-1} \leq R_o$ or $S^*_{N-1} < R_i$ or $S^*_{N-1} > R_o$ prevails, and correspondingly in equation (49) three different expressions must be introduced, by no means affects the solution procedure. It is merely necessary, to begin with, to obtain with the aid of equation (23) an exact idea of the dependence of the magnitude of S^*_{N-1} on the chosen space point r_j, ϕ_{o1}.

5. The Integral Equation of the Airflow Boundary Condition

In Nos. 2 and 3 we have dealt with the induced velocities corresponding to the vortex model at the airfoil blade of the rotor. If we insert these into the expressions (3) to (6), (8) to (10), as well as (19), (24), and (29) in the airflow boundary condition (12) at the rotor airfoil, then an integral equation of the general type (42) results, as was discussed in No. 4b. At this point it is superfluous to once more write down the individual extensive core portion. They can be inferred directly from the above-mentioned formulas. Moreover, the fixed cores must thereby be combined in a clear, condensed expression, as in equation (42). It is merely to be further observed that the integral equation originating from (12) differs thereby from equation (42),

that in it two characteristic hypothetical terms occur through unrolled transverse vortices, i.e., tip vortex and root vortex. Thus the method of solution does not change of itself. The formulation (43) is simply to be expanded into (with $\phi_\alpha = \phi_o - 1/2\alpha_o - 1/2\alpha_i$):

$$\Gamma(s,\varphi_o) \equiv \Gamma(\tau,\varphi_o) = \Gamma_m(\varphi_\alpha - \tfrac{2\pi}{N} + \psi_o^{(o)})\, f_o(\varphi_\alpha,\alpha_o)\, \gamma_o(\tau,\varphi_\alpha) + \Gamma_m(\varphi_\alpha - \tfrac{2\pi}{N} + \psi_o^{(i)})\, f_i(\varphi_\alpha,\alpha_i)\, \gamma_i(\tau,\varphi_\alpha) + \Gamma_*(\tau,\varphi_o)\,; \quad \text{(with } \varphi_\alpha = \varphi_o - \tfrac{1}{2}\alpha_o - \tfrac{1}{2}\alpha_i\text{)} \tag{54}$$

with γ_o and γ_i as solutions of the integral equations (see (31), (34)):

$$\frac{\tau - \tau_o(\varphi_o,\alpha_o)}{\ell_*^2 + [\tau - \tau_o(\varphi_o,\alpha_o)]^2} = \frac{1}{2\pi}\int_{R_i}^{R_o}\left[\frac{\partial\gamma_o(s,\varphi_o)}{\partial s} + \frac{1}{2s}\gamma_o(s,\varphi_o)\right]\frac{ds}{\tau - s}\,; \quad (\alpha \approx \tfrac{1}{2}\alpha_o + \tfrac{1}{2}\alpha_i) \tag{55_o}$$

$$-\frac{\tau - \tau_i(\varphi_o,\alpha_i)}{\ell_*^2 + [\tau - \tau_i(\varphi_o,\alpha_i)]^2} = \frac{1}{2\pi}\int_{R_i}^{R_o}\left[\frac{\partial\gamma_i(s,\varphi_o)}{\partial s} + \frac{1}{2s}\gamma_i(s,\varphi_o)\right]\frac{ds}{\tau - s}\,. \tag{55_i}$$

/27

In place of equation (46) there now appears with $\psi_o^{(o)} \approx 2\pi/N$ and $\psi_o^{(i)} \approx 2\pi/N$:

$$\Gamma_m(\varphi_o) = \Gamma_*(\tfrac{\pi}{2},\varphi_o)\left[1 - f_o(\varphi_o,\alpha_o)\,\gamma_o(\tfrac{\pi}{2},\varphi_o) - f_i(\varphi_o,\alpha_i)\,\gamma_i(\tfrac{\pi}{2},\varphi_o)\right]^{-1}. \tag{56}$$

Since $\gamma_o(\pi/2, \phi_o) < 0$ and $\gamma_i(\pi/2, \phi_o) < 0$, the denominator from (56) always remains positive.

Neglecting even the little supplement mentioned, the resolution of the integral equation for the boundary edge follows exactly as that from equation (42) or (52). Consequently at this point we cannot go into further details.

6. Calculation of the Airfoil Force

The momentary axial force K_x and peripheral force K_ϕ (per unit of length in the radial direction) produced in the rotor airfoil are calculated as usual

with the help of the Kutta-Joukowski principle [1][7]. From both of these force components mentioned, the force in other directions can be put together.

As is well known from the Jutta-Joukowski Principle, the induced velocities from the vortex system are necessary at the supporting line of the airfoil space point ($x = 0$, $y = r \cos \phi_o$, $z = r \sin \phi_o$) (Figure 3).

In Nos. 2 and 3 of the present work we have investigated the induced velocities from the vortex system of the blades along the 3/4-line ($x = 0$, $y = r \cos (\phi_o + \alpha)$, $z = r \sin (\phi_o + \alpha)$) of the airfoil space point. There now remains to be proved to what extent these constructions can be applied even for the calculation of induced velocities at the site of the supporting line ($x = 0$, $y = r \cos \phi_o$, $z = r \sin \phi_o$) and, if the occasion arises, which modifications are necessary.

A second inspection of careful consideration of the results from Nos. 2 and 3 shows:

For ω_Γ, according to equation (3) α must be equal to zero, and the summand $n = 0$ be omitted.

In $\sum_{n-1}^{N-2} \omega_Q^{(n)}$ and $\sum_{n-1}^{N-1} \omega_L^{(m)}$ according to formulas (5) and (9) α need only be equated to zero.

With $\omega_Q^{(N-1)}$ and $\omega_L^{(N-1)}$, $\alpha = \alpha_0 = \alpha_i = \alpha_o' = \alpha_i' = 0$ is to be added in the integrands in formulas (4), (10), (24), and (29). The latter, however, does not hold at the integration limits or χ-values, which must remain preserved unchanged. Besides equation (7) and (11), (21) to (23), and (25) to (28), $\alpha = \alpha_o = \alpha_i = \alpha_o' = \alpha_i' = 0$ is to be taken as fundamental. /28

The above is also valid at times for the second integral of $\omega_Q^{(o)}$ and $\omega_L^{(o)}$ in formulas (6) and (8). Both first integrals of the induced velocity, which were previously investigated only in No. 3a under the reduced term $(u_Q^{(o)} + u_L^{(o)})$ along the 3/4-line of the airfoil space point, now require for their calculation at the site of the supporting line a modified construction in relation to No. 3a. Consequently, use should thereby also be made now of the simplified approximation $k_* = 0$.

[7] n.b., p. 162-164.

The necessity for this modified construction is thus subject, as is known, to these induced velocity constituents becoming singular at a systematic substitution of the coordinates $x = 0$, $y = r \cos \phi_o$, $z = r \sin \phi_o$ of the supporting line [1]. By validity calculation, according to the Kutta-Joukowski principle, for the applied expressions, they are much more to be acquired for the avoidance of a singularity through a boundary transition from the supporting-surface theory. Therefore one proceeds not from angle ϕ_o of the supporting vortex line, but from

$$\phi_o + \lim_{\chi \to o} \chi$$

with χ as the angle coordinate in the direction of the airfoil depth. $\chi_V(r)$ denotes the leading edge and $\chi_H(r)$ the trailing edge [1] (Figure 7).

Inasmuch as integrands result for $\chi \to 0$ which, as Cauchy principle values, ln-singularities, or even as constant functions, are integratable, $\chi = 0$ can be applied directly into the constituents in question. The critical terms for which $\chi \to 0$, integrals over s that fail to approach a limit would be formed, can be interpreted by the integral formula [1]:

$$\lim_{\chi \to o} \int_{R_i}^{R_o} \frac{ds}{\sqrt{(r-s)^2 + r^2\chi^2}} = \lim_{\chi \to o} \ln \frac{4(R_o - r)(r - R_i)}{r^2\chi^2} = 2 \ln \frac{4\sqrt{(R_o - r)(r - R_i)}}{r[\chi_H(r) - \chi_V(r)]} + 2 \ln 2 + 1 . \tag{57}$$

For the rationale of this method and the derivation of (57), reference must be made to the literature mentioned.

We even obtain with this developed method chiefly from equation (6) and (8) for $x = 0$, $y = r \cos(\phi_o + \chi)$, $z = r \sin(\phi_o + \chi)$ and $\chi \to 0$: /29

$$\text{Term } (u_Q^{(o)} + u_L^{(o)}) = \frac{1}{4\pi} \lim_{\chi \to o} \int_{R_i}^{R_o} \int_{o}^{2\alpha} [r^2 + s^2 - 2rs \cos(\chi - \psi)]^{-3/2} \cdot \left\{ \frac{\partial \Gamma(s, \varphi_o + \psi)}{\partial s} (rs \cos(\chi - \psi) - s^2) + \frac{\partial \Gamma(s, \varphi_o + \psi)}{\partial \varphi_o} r \sin(\chi - \psi) \right\} d\psi \, ds . \tag{58}$$

A simple investigation of the integrals in the vicinity of the locus $s = r$ and $\psi = 0$ shows that $\chi = 0$ needs to be inserted directly in the $(u_Q^{(o)})$ term. The distinct expression results:

$$\text{Term}\,(u_Q^{(o)}) = \frac{1}{4\pi}\int_{R_i}^{R_o}\int_{0}^{2\alpha}\Big\{\big[r^2+s^2-2rs\cos\psi\big]^{-3/2}\cdot\frac{\partial\Gamma(s,\varphi_0+\psi)}{\partial s}(rs\cos\psi-s^2)-$$
$$-\big[(r-s)^2+rs\psi^2\big]^{-3/2}\cdot\frac{\partial\Gamma(s,\varphi_0)}{\partial s}s\Big(r-s-r\frac{\psi^2}{2}\Big)\Big\}d\psi\,ds+ \qquad (59)$$
$$+\frac{1}{4\pi}\int_{R_i}^{R_o}\frac{\partial\Gamma(s,\varphi_0)}{\partial s}\left[\frac{2\alpha s}{\sqrt{4\alpha^2rs+(r-s)^2}}\frac{1}{r-s}+\frac{\alpha}{\sqrt{4\alpha^2rs+(r-s)^2}}-\right.$$
$$\left.-\frac{1}{2\sqrt{rs}}\ln\frac{2\alpha\sqrt{rs}+\sqrt{4\alpha^2rs+(r-s)^2}}{|r-s|}\right]ds\,.$$

The integrand of the double integrals in formula (59) for $s = r$ and $\psi = 0$ is resticted. Through series expansion with respect to ψ, it is shown that the value

$$\lim_{\psi\to 0}\lim_{s\to r}\left(\frac{\psi s(r-s)-\frac{1}{2}\psi^3rs}{\sqrt{(r-s)^2+rs\psi^2}^3}\right)\frac{\partial^2\Gamma(s,\varphi_0)}{\partial s\,\partial\varphi_0}=-\frac{1}{2r}\frac{\partial^2\Gamma(r,\varphi_0)}{\partial r\,\partial\varphi_0}$$

has to be coordinated to it. The integrand of the second simple integral over s in (59) includes the known, easily usable singularities of a Cauchy principle value, and of a logarithm. For the $(u_L^{(o)})$ term, we obtain, with the substitution of $\psi - \chi = \eta$ in the limiting process, the expression:

$$\text{Term}\,(u_L^{(o)}) = -\frac{1}{4\pi}\int_{R_i}^{R_o}\int_{0}^{2\alpha}\Big\{\big[r^2+s^2-2rs\cos\eta\big]^{-3/2}\cdot\frac{\partial\Gamma(s,\varphi_0+\eta)}{\partial\varphi_0}r\sin\eta-$$
$$-\big[(r-s)^2+rs\eta^2\big]^{-3/2}\cdot r\eta\left(\frac{\partial\Gamma(s,\varphi_0)}{\partial\varphi_0}+\eta\frac{\partial^2\Gamma(s,\varphi_0)}{\partial\varphi_0^2}\right)\Big\}d\eta\,ds+$$
$$+\frac{1}{4\pi}\int_{R_i}^{R_o}\frac{\partial\Gamma(s,\varphi_0)}{\partial\varphi_0}\frac{1}{s}\frac{ds}{\sqrt{4\alpha^2rs+(r-s)^2}}+\frac{1}{4\pi}\int_{R_i}^{R_o}\frac{\partial^2\Gamma(s,\varphi_0)}{\partial\varphi_0^2}\left[\frac{2\alpha}{s\sqrt{4\alpha^2rs+(r-s)^2}}-\right.$$
$$\left.-\frac{1}{\sqrt{rs^3}}\ln\frac{2\alpha\sqrt{rs}+\sqrt{4\alpha^2rs+(r-s)^2}}{|r-s|}\right]ds-$$
$$-\frac{1}{4\pi}\lim_{\chi\to 0}\int_{R_i}^{R_o}\frac{\partial\Gamma(s,\varphi_0)}{\partial\varphi_0}\frac{1}{s}\frac{ds}{\sqrt{\chi^2rs+(r-s)^2}}\,. \qquad (60)$$

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The last integral in formula (60) is interpreted with the aid of equation (57). For this there is obtained:

$$\lim_{\chi\to 0}\int_{R_i}^{R_0}\frac{\partial\Gamma(s,\varphi_0)}{\partial\varphi_0}\frac{1}{s}\frac{ds}{\sqrt{\chi^2 r s+(r-s)^2}}=\int_{R_i}^{R_0}\left(\frac{\partial\Gamma(s,\varphi_0)}{\partial\varphi_0}\frac{1}{s}-\frac{\partial\Gamma(r,\varphi_0)}{\partial\varphi_0}\frac{1}{r}\right)\frac{ds}{|r-s|}$$
$$+\frac{1}{r}\frac{\partial\Gamma(r,\varphi_0)}{\partial\varphi_0}\left[2\ln\frac{4\sqrt{(R_0-r)(r-R_i)}}{r[\chi_H(r)-\chi_V(r)]}+2\ln 2+1\right]. \qquad (61)$$

The integral on the right side in (61) has a varying integrand at s = r. The arithmetic average, namely zero, is to be associated with it.

There now remains, in addition, to determine the limiting value of the integrand at the double integral in formula (60) for s = r and η = 0. By series expansion with respect to η this results in

$$\lim_{\eta\to 0}\lim_{s\to r} r\left[-\frac{1}{6}\frac{\eta^3}{\sqrt{(r-s)^2+rs\eta^2}^3}+\frac{1}{8}\frac{rs\eta^5}{\sqrt{(r-s)^2+rs\eta^2}^5}\right]\frac{\partial\Gamma(s,\varphi_0)}{\partial\varphi_0}+$$
$$+\frac{1}{2}\lim_{\eta\to 0}\lim_{s\to r} r\frac{\eta^3}{\sqrt{(r-s)^2+rs\eta^2}^3}\frac{\partial^3\Gamma(s,\varphi_0)}{\partial\varphi_0^3}=-\frac{1}{24}\frac{1}{r^2}\frac{\partial\Gamma(r,\varphi_0)}{\partial\varphi_0}+\frac{1}{2r^2}\frac{\partial^3\Gamma(r,\varphi_0)}{\partial\varphi_0^3}.$$

With this the investigation of the induced velocities required for force calculation according to the Kutta-Joukowski principle is concluded.

7. Integral Formulas

In the preceding paragraphs several important integral formulas were used which should be compiled and shown in the following.

a) By means of the residue method we examine the $Z = e^{i\tau}$ over the unit circle of the complexed Z-gradient of the fully evaluated integral

$$I_n=\frac{1}{\pi}\int_0^{\pi}\frac{\cos n\tau\, d\tau}{(\cos\tau-\cos\vartheta)[\ell^2+(\cos\tau-\Xi)^2]}=$$
$$=\frac{4}{\pi i}\oint_{(UC)}\frac{Z^{n-1}dZ}{(Z+\frac{1}{Z}-2\cos\vartheta)(Z+\frac{1}{Z}-2\Xi-2i\ell)(Z+\frac{1}{Z}-2\Xi+2i\ell)}.$$

The zero points in the denominator of the integrand lie at $Z_1 = e^{i\vartheta}$, /31

$$Z_2 = \Xi + i\ell + \sqrt{\Xi^2 - 1 - \ell^2 + 2i\ell\Xi}\,, \quad Z_3 = \Xi + i\ell - \sqrt{\Xi^2 - 1 - \ell^2 + 2i\ell\Xi}\,,$$

as well as at the conjugated complex values.

It is considered that the following are valid:

$$Z_1 \bar{Z}_1 = 1\,; \quad Z_2 Z_3 = 1\,; \quad Z_1 + \bar{Z}_1 = 2\cos\vartheta\,;$$

$$Z_2 + \frac{1}{Z_2} = Z_3 + \frac{1}{Z_3} = 2\Xi + 2i\ell.$$

For application in the rotor airflow, it is to be seen that $b^2 \ll 1$. In the evaluation of the integral I_n and the discussion of the zero points Z_2, Z_3 the complex Z-gradient along the positive real axis should be opened for fixing the principle values of the root[6]. By simple series expansion with consideration of various possibilities, $|\Xi^2 - 1| \gg b^2$ with $\Xi \approx \pm 1$, or $|\Xi^2 - 1| \gg b^2$ with $\Xi > 1$, $\Xi < -1$, $\Xi > -1$ or $\Xi^2 - 1 = b^2$ with $\Xi \approx \pm (1 + b^2/2)$ it is shown that $|Z_2| > 1$ and thus $|Z_3|$ always remains < 1. With the aid of the residue method we then obtain:

$$I_n = \frac{1}{\ell^2 + (\cos\vartheta - \Xi)^2}\left[\frac{\sin n\vartheta}{\sin\vartheta} + \frac{i}{2\ell}\,\frac{\Xi - \cos\vartheta - i\ell}{\sqrt{(\Xi + i\ell)^2 - 1}}\left[\Xi + i\ell - \sqrt{(\Xi + i\ell)^2 - 1}\right]^n - \frac{i}{2\ell}\,\frac{\Xi - \cos\vartheta + i\ell}{\sqrt{(\Xi - i\ell)^2 - 1}}\left[\Xi - i\ell - \sqrt{(\Xi - i\ell)^2 - 1}\right]^n\right]. \tag{62}$$

There especially follows from formula (62):

$$I_0 = \frac{1}{\ell}\,\frac{(\Xi - \cos\vartheta)\sin\frac{\phi}{2} + \ell\cos\frac{\phi}{2}}{[\ell^2 + (\cos\vartheta - \Xi)^2]\sqrt[4]{(\Xi^2 - 1 - \ell^2)^2 + 4\Xi^2\ell^2}}\,;$$

$$I_2 = -\frac{2}{\ell}\,\frac{\sqrt[4]{(\Xi^2 - 1 - \ell^2)^2 + 4\Xi^2\ell^2}}{\ell^2 + (\cos\vartheta - \Xi)^2}\left[(\Xi - \cos\vartheta)\sin\frac{\phi}{2} - \ell\cos\frac{\phi}{2}\right] + \frac{1}{\ell}\,\frac{(\Xi - \cos\vartheta)\sin\frac{\phi}{2} + \ell\cos\frac{\phi}{2}}{[\ell^2 + (\cos\vartheta - \Xi)^2]\sqrt[4]{(\Xi^2 - 1 - \ell^2)^2 + 4\Xi^2\ell^2}}\,; \tag{63}$$

with the arrangement: $\Xi^2 - 1 - \ell^2 + 2i\ell\Xi \equiv \sqrt{(\Xi^2 - 1 - \ell^2)^2 + 4\Xi^2\ell^2}\cdot e^{i\phi}$.

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[6] One can also cut the Z-plane along the solid negative axis and then come under consideration of the relationship naturally valid for the equal ultimate result for I_n.

The combination $\frac{\sqrt[4]{\pi}}{\sin\frac{\Phi}{2}} I_0 + \frac{1}{2}\frac{I_0 - I_2}{\sqrt[4]{\pi}\cdot\sin\frac{\Phi}{2}}$ then yields the sought integral needed for the rotor airflow:

$$\frac{\ell}{\sqrt{2}\pi}\left[1-\Xi^2+\ell^2+\sqrt{(\Xi^2-1-\ell^2)^2+4\Xi^2\ell^2}\right]^{-1/2}\cdot \int_0^{\pi} \frac{\sqrt{(\Xi^2-1-\ell^2)^2+4\Xi^2\ell^2}+\sin^2\tau}{(\cos\tau-\cos\vartheta)\left[\ell^2+(\cos\tau-\Xi)^2\right]}\,d\tau = \frac{\Xi-\cos\vartheta}{\ell^2+(\cos\vartheta-\Xi)^2}\ ; \tag{64}$$

since

$$\sin\frac{\Phi}{2} = \frac{\sqrt{1-\Xi^2+\ell^2+\sqrt{(\Xi^2-1-\ell^2)^2+4\Xi^2\ell^2}}}{\sqrt{2}\sqrt[4]{(\Xi^2-1-\ell^2)^2+4\Xi^2\ell^2}}\ , \quad (0<\tfrac{\Phi}{2}<\pi)$$

exists.

In an analogous manner both integral formulas are demonstrated with the aid of the residue method:

$$\frac{1}{\pi}\int_0^{\pi}\frac{2\ell+\sin^2\tau}{\ell^2+(\cos\tau-1)^2}\,d\tau = -1+2\,\frac{(1+\frac{\ell}{2})\cos\frac{\Phi_*}{2}+\sin\frac{\Phi_*}{2}}{\sqrt[4]{4\ell^2+\ell^4}} \approx$$

$$\approx -1+\frac{2}{\sqrt{\ell}}+\frac{1}{2}\sqrt{\ell}+O(\ell)\ ;$$

$$\frac{1}{\pi}\int_0^{\pi}\cos\tau\,\frac{2\ell^2+\sin^2\tau}{\ell^2+(\cos\tau-1)^2}\,d\tau = 2-\frac{(2+3\ell)\cos\frac{\Phi_*}{2}+(2-2\ell-\ell^2)\sin\frac{\Phi_*}{2}}{\sqrt[4]{4\ell^2+\ell^4}} \approx \tag{65}$$

$$\approx 2-\frac{2}{\sqrt{\ell}}-\frac{1}{2}\sqrt{\ell}+O(\ell)\ .$$

Therefore

$$2i\ell+\ell^2 = \sqrt{4\ell^2+\ell^4}\cdot e^{i\Phi_*}\ ;\quad \operatorname{tg}\Phi_* = \frac{2}{\ell}\ .$$

Finally the complex Z-gradient is obtained by integration over the unit circle by the residue method:

$$\frac{1}{\pi}\int_0^{\pi}\frac{\cos\nu\tau\,d\tau}{\cos\tau-\cos\vartheta} = \frac{\sin\nu\vartheta}{\sin\vartheta}\ ; \qquad (0\le\vartheta\le\pi) \tag{66}$$

$$\frac{1}{\pi}\int_0^{\pi}\frac{\cos\nu\tau\,d\tau}{\cos\tau-\operatorname{Cof}\vartheta^*} = -\frac{e^{-\nu\vartheta^*}}{\operatorname{Sin}\vartheta^*}\ ; \qquad (\vartheta^*>0) \tag{67}$$

$$\frac{1}{\pi}\int_0^{\pi}\frac{\cos\nu\tau\,d\tau}{\cos\tau+\cosh\vartheta^*}=(-1)^{\nu}\frac{e^{-\nu\vartheta^*}}{\sinh\vartheta^*}\quad;\quad(\vartheta^*>0)\,. \tag{68}$$

b) From integral tables the three integrals

$$\int[A\chi^2+2B\chi+C]^{-3/2}\,\chi^m\,d\chi\qquad(m=0,1,2)$$

are known [1].

Through differentiation of A, B, or C the further integral is obtained:

$$\int\frac{d\chi}{\sqrt{A\chi^2+2B\chi+C}^5}=\frac{1}{3}\frac{A\chi+B}{AC-B^2}\frac{1}{\sqrt{A\chi^2+2B\chi+C}^3}+\frac{2}{3}\frac{A\chi+B}{(AC-B^2)^2}\frac{A}{\sqrt{A\chi^2+2B\chi+C}}\,; \tag{69}$$

$$\int\frac{\chi\,d\chi}{\sqrt{A\chi^2+2B\chi+C}^5}=-\frac{1}{3}\frac{B\chi+C}{AC-B^2}\frac{1}{\sqrt{A\chi^2+2B\chi+C}^3}-\frac{2}{3}\frac{A\chi+B}{(AC-B^2)^2}\frac{B}{\sqrt{A\chi^2+2B\chi+C}}\,; \tag{70}$$

$$\int\frac{\chi^2\,d\chi}{\sqrt{A\chi^2+2B\chi+C}^5}=\frac{1}{3}\frac{A\chi+B}{AC-B^2}\frac{\chi^2}{\sqrt{A\chi^2+2B\chi+C}^3}+\frac{2}{3}\frac{B\chi+C}{(AC-B^2)^2}\frac{B}{\sqrt{A\chi^2+2B\chi+C}}\,; \tag{71}$$

$$\int\frac{\chi^3\,d\chi}{\sqrt{A\chi^2+2B\chi+C}^5}=-\frac{1}{3}\frac{B\chi+C}{AC-B^2}\frac{\chi^2}{\sqrt{A\chi^2+2B\chi+C}^3}-\frac{2}{3}\frac{B\chi+C}{(AC-B^2)^2}\frac{C}{\sqrt{A\chi^2+2B\chi+C}}\,; \tag{72}$$

$$\begin{aligned}\int\frac{\chi^4\,d\chi}{\sqrt{A\chi^2+2B\chi+C}^5}&=\frac{1}{\sqrt{A}^5}\ln\left|A\chi+B+\sqrt{A}\sqrt{A\chi^2+2B\chi+C}\right|+\\&+\frac{2}{3}\frac{C\chi}{A(AC-B^2)}\frac{1}{\sqrt{A\chi^2+2B\chi+C}}+\frac{2}{3}\frac{(2AC-B^2)(2B^2\chi-AC\chi+BC)}{A^2(AC-B^2)^2\sqrt{A\chi^2+2B\chi+C}}+\\&+\frac{1}{3}\frac{\chi^2}{A(AC-B^2)}\frac{(2B^2-AC)\chi+BC}{\sqrt{A\chi^2+2B\chi+C}^3}-\\&-\frac{2}{3}\frac{1}{A^2\sqrt{A\chi^2+2B\chi+C}}\frac{\sqrt{A}\sqrt{A\chi^2+2B\chi+C}\cdot\chi+A\chi^2+B\chi+\frac{1}{2}C}{A\chi+B+\sqrt{A}\sqrt{A\chi^2+2B\chi+C}}\,.\end{aligned} \tag{73}$$

If $B\sigma$ is substituted for B, and $C\sigma^2$ for C in the above integral formulas, then the following important relationships result:

$$\lim_{\sigma\to 0}\left[\int_{-\alpha}^{\alpha}\frac{\sigma\,dx}{\sqrt{Ax^2+2Bx\sigma+C\sigma^2}^3}-\frac{2\sqrt{A}}{\sigma(AC-B^2)}\right]=0\,; \tag{74}$$

$$\lim_{\sigma\to 0}\left[\int_{-\alpha}^{\alpha}\frac{x\,dx}{\sqrt{Ax^2+2Bx\sigma+C\sigma^2}^3}+\frac{2B}{\sigma(AC-B^2)\sqrt{A}}\right]=0\,; \tag{75}$$

$$\lim_{\sigma\to 0}\left[\int_{-\alpha}^{\alpha}\frac{x^2\,dx}{\sqrt{Ax^2+2Bx\sigma+C\sigma^2}^3}+\frac{1}{\sqrt{A}^3}\ln\frac{\sigma^2(AC-B^2)}{4A^2\alpha^2}\right]=\frac{2(2B^2-AC)}{\sqrt{A}^3(AC-B^2)}\,; \tag{76}$$

$$\lim_{\sigma\to 0}\left[\int_{-\alpha}^{\alpha}\frac{\sigma^4\,dx}{\sqrt{Ax^2+2Bx\sigma+C\sigma^2}^5}\right]=\frac{4}{3}\frac{\sqrt{A}^3}{(AC-B^2)^2}\,; \tag{77}$$

$$\lim_{\sigma\to 0}\left[\int_{-\alpha}^{\alpha}\frac{x\sigma^3\,dx}{\sqrt{Ax^2+2Bx\sigma+C\sigma^2}^5}\right]=-\frac{4}{3}\frac{B\sqrt{A}}{(AC-B^2)^2}\,; \tag{78}$$

$$\lim_{\sigma\to 0}\left[\int_{-\alpha}^{\alpha}\frac{x^2\sigma^2\,dx}{\sqrt{Ax^2+2Bx\sigma+C\sigma^2}^5}\right]=\frac{4}{3}\frac{B^2}{(AC-B^2)^2\sqrt{A}}+\frac{2}{3}\frac{1}{(AC-B^2)\sqrt{A}^3}\,; \tag{79}$$

$$\lim_{\sigma\to 0}\left[\int_{-\alpha}^{\alpha}\frac{\sigma x^3\,dx}{\sqrt{Ax^2+2Bx\sigma+C\sigma^2}^5}\right]=-\frac{4}{3}\frac{BC}{(AC-B^2)^2\sqrt{A}}-\frac{2}{3}\frac{B}{(AC-B^2)\sqrt{A}^3}\,; \tag{80}$$

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$$\lim_{\sigma\to 0}\left[\int_{-\alpha}^{\alpha}\frac{x^4\,dx}{\sqrt{Ax^2+2Bx\sigma+C\sigma^2}^5}+\frac{1}{\sqrt{A}^5}\ln\frac{\sigma^2(AC-B^2)}{4A^2\alpha^2}\right]=$$

$$=\frac{4}{3}\frac{2B^2}{(AC-B^2)\sqrt{A}^5}+\frac{4}{3}\frac{(2AC-B^2)(2B^2-AC)}{(AC-B^2)^2\sqrt{A}^5}\,. \tag{81}$$

For a continuous function $\Gamma(s)$ that can be differentiated, the following are valid, assuming that $\Gamma(R_i) = \Gamma(R_o) = 0$:

$$\frac{1}{2}\int_{R_i}^{R_o} \frac{d\Gamma(s)}{ds} \ln \frac{(s-z)^2}{R_o^2} ds = -\int_{R_i}^{R_o} \frac{\Gamma(s)}{s-z} ds \; ; \tag{82}$$

$$\int_{R_i}^{R_o} \Gamma(s) \ln \frac{(s-z)^2}{R_o^2} ds = -\int_{R_i}^{R_o} \frac{d\Gamma(s)}{ds} \left[\ln \frac{(s-z)^2}{R_o^2} - 2 \right] (s-z)\, ds \; . \tag{83}$$

APPENDIX

8. The Acoustic Pressure Field of a Rotor

In many cases the acoustic pressure field radiated from a rotor is of interest. Among other things, a considerable noise originates when a rotor blade penetrates a free tip vortex or root vortex, a process which we have dealt with in this work. Also, when the airflow velocities at the rotor airfoil are situated so far below sonic velocity that they have to be considered incompressible with respect to boundary conditions[8] (as occurs in Nos. 2 to 6 of the submitted investigation), the compressibility of air as calculated for the radiated acoustic distant field must be kept in mind[9].

By development of the appropriate formula we take as the basis the usual linearization assumption of airflow acoustics. A linear pressure-density relation and neglect of the square, respectively, are products of all the velocities induced from the singularities system of the rotor[9].

Between the velocity potential of this singularity system (dipole, origins) and the pressure field there exists then the relationship:

$$\frac{1}{\rho_0} p - \frac{1}{\rho_0} p_0 = -\frac{\partial \phi}{\partial t} - u_0 \frac{\partial \phi}{\partial x} - w_0 \frac{\partial \phi}{\partial z} = \omega \frac{\partial \phi}{\partial \varphi_0} - u_0 \frac{\partial \phi}{\partial x} - w_0 \frac{\partial \phi}{\partial z} ,$$

with $\omega dt = -d\phi_0$, and the velocity potential ϕ satisfies a generalized wave equation:

$$\left(1 - \frac{u_0^2}{c_0^2}\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \left(1 - \frac{w_0^2}{c_0^2}\right) \frac{\partial^2 \phi}{\partial z^2} - 2 \frac{u_0 w_0}{c_0^2} \frac{\partial^2 \phi}{\partial x \partial z} -$$

$$- \frac{2 u_0}{c_0^2} \frac{\partial^2 \phi}{\partial x \partial t} - \frac{2 w_0}{c_0^2} \frac{\partial^2 \phi}{\partial z \partial t} - \frac{1}{c_0^2} \frac{\partial^2 \phi}{\partial t^2} = f(x, y, z, t) ,$$

where $f(x, y, z, t) = \sum_{(n)} Q(u_n, t)\, \delta(u - u_n)$ is nonzero at the point of source singularities. In equation (85) C_0 is taken as the constan velocity of sound throughout the entire airflow range, and u_0, w_0, are the airflow velocities in

[8] As can be demonstrated by series expansion at the reciprocal wave velocity $1/c_0$ the formulas distinguish between the compressible and the incompressible theory only by the $\sim 1/c_0$ term. By this means, the range of validity of the incompressible theory for the near field becomes favorable (see [1], pp. 117, 119).

[9] n.b., Ch. II E.

the airflow velocities in the x and y directions relative to the helicopter (Figure 1). From the pressure field ρ, a wave equation of type (85) suffices according to (84). Indeed, with the non-homogeneous right side

$$-\left(\frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} + w_0 \frac{\partial}{\partial z}\right) \rho_0 \sum_{(n)} Q(4_n, t) \cdot \delta(4 - 4_n) \; .$$

equation (85) thereby differs from the wave equations appearing in the normal propeller theory, wherein two principle airflow velocities appear in the x and y directions. We therefore transform (85) to a new coordinate system $\bar{x}$, y, Z, i.e., we set up:

$$\begin{aligned} x &= \bar{x}\cos\lambda - Z\sin\lambda \; ; \quad z = \bar{x}\sin\lambda + Z\cos\lambda \\ u_0 &= U\cos\lambda \; ; \quad w_0 = U\sin\lambda \; . \end{aligned} \tag{86}$$

With the transformed potential

$$\bar{\Phi}(\bar{x}, y, Z, t) = \bar{\Phi}(x\cos\lambda + z\sin\lambda, y, -x\sin\lambda + z\cos\lambda, t) \equiv \Phi(x, y, z, t) \tag{87}$$

equation (85) then takes the simpler form

$$\begin{aligned} &\left(1 - \frac{U^2}{c_0^2}\right)\frac{\partial^2\bar{\Phi}}{\partial\bar{x}^2} + \frac{\partial^2\bar{\Phi}}{\partial y^2} + \frac{\partial^2\bar{\Phi}}{\partial Z^2} - \frac{2U}{c_0^2}\frac{\partial^2\bar{\Phi}}{\partial\bar{x}\partial t} - \frac{1}{c_0^2}\frac{\partial^2\bar{\Phi}}{\partial t^2} = \\ &= f(\bar{x}\cos\lambda - Z\sin\lambda, y, \bar{x}\sin\lambda + Z\cos\lambda, t) \equiv F(\bar{x}, y, Z, t) \; , \end{aligned} \tag{88}$$

The solution of (88) is known, and appears as [1][10]:

$$\bar{\Phi}(\bar{x}, y, Z, t) = -\frac{1}{4\pi}\iiint_{(\mathcal{X})} \frac{1}{D} F\left(\bar{x}', y', Z', t + \frac{U}{\beta^2 c_0^2}(\bar{x} - \bar{x}') - \frac{D}{\beta^2 c_0}\right) d\bar{x}'dy'dZ'$$

$$\text{with} \quad D = \left[(\bar{x} - \bar{x}')^2 + \beta^2(y - y')^2 + (Z - Z')^2\beta^2\right]^{1/2} \; ; \tag{89}$$

$$\beta^2 = 1 - U^2/c_0^2 = 1 - (u_0^2 + w_0^2)/c_0^2 \; .$$

[10] n.b., Ch. II E.

In equation (89) integration over the range (L) is to be extended so that F differs from zero. From (89) there follows further:

$$\phi(x,y,z,t) = -\frac{1}{4\pi}\iiint_{(\mathcal{L})} \frac{1}{D} f\left(x',y',z',t+\frac{u_0}{\beta^2 c_0^2}(x-x')+\frac{w_0}{\beta^2 c_0^2}(z-z')-\frac{D}{\beta^2 c_0}\right)\cdot \\ \cdot dx'dy'dz'; \qquad (90)$$

$$\text{with } D = \left[(x-x')^2\left(1-\frac{w_0^2}{c_0^2}\right)+(z-z')^2\left(1-\frac{u_0^2}{c_0^2}\right)+\beta^2(y-y')^2+\right. \\ \left.+2(x-x')(z-z')\frac{u_0 w_0}{c_0^2}\right]^{1/2},$$

or in cylindrical coordinates with $\omega dt = d\phi_0$:

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$$\phi(x,r,\varphi,\varphi_0) = -\frac{1}{4\pi}\iiint_{(\mathcal{L})} \frac{1}{D} f\left(x',r',\varphi',\varphi_0-\frac{\omega u_0}{\beta^2 c_0^2}(x-x')-\right. \\ \left.-\frac{\omega w_0}{\beta^2 c_0^2}(r\sin\varphi - r'\sin\varphi')+\frac{\omega D}{\beta^2 c_0}\right) r'dr'd\varphi'dx'; \qquad (91)$$

$$\text{with } D = \left[(x-x')^2\left(1-\frac{w_0^2}{c_0^2}\right)+\left(1-\frac{u_0^2}{c_0^2}\right)\left(r^2+r'^2-2rr'\cos(\varphi-\varphi')\right)-\right. \\ \left.-\frac{w_0^2}{c_0^2}(r\cos\varphi - r'\cos\varphi')^2+2\frac{u_0 w_0}{c_0^2}(x-x')(r\sin\varphi - r'\sin\varphi')\right]^{1/2}.$$

From equation (91) can be obtained the fundamental solution ϕ_0 required for handling the rotor airflow, namely, a source element of force $g(s, x, \phi_0)$ rotating with an angle of velocity ω and momentarily situated at the point where $x = k_1\chi$, $r = s$, $\phi = \phi_0 + \chi$. In this special case,

$$f(x',r',\varphi',\varphi_0) = q(s,\chi,\varphi_0)\,\delta(x-k_1\chi)\,\frac{1}{r'}\,\delta(r'-s)\,\delta(\varphi'-\varphi_0-\chi)$$

and with a simple intermediate computation (see [13]) there is obtained from equation (91):

$$\phi(x,r,\varphi,\varphi_0) = -\frac{1}{4\pi}\iiint_{(\mathcal{L})} \frac{1}{D} f\left(x',r',\varphi',\varphi_0-\frac{\omega u_0}{\beta^2 c_0^2}(x-x')-\right. \\ \left.-\frac{\omega w_0}{\beta^2 c_0^2}(r\sin\varphi - r'\sin\varphi')+\frac{\omega D}{\beta^2 c_0}\right) r'dr'd\varphi'dx';$$

$$\text{mit: } D = \left[(x-x')^2\left(1-\frac{w_0^2}{c_0^2}\right)+\left(1-\frac{u_0^2}{c_0^2}\right)\left(r^2+r'^2-2rr'\cos(\varphi-\varphi')\right)-\right. \\ \left.-\frac{w_0^2}{c_0^2}(r\cos\varphi - r'\cos\varphi')^2+2\frac{u_0 w_0}{c_0^2}(x-x')(r\sin\varphi - r'\sin\varphi')\right]^{1/2}.$$

$$\Phi_0(x,r,\varphi,\varphi_0) = -\frac{1}{4\pi} q\left(s,\chi,\varphi_0 - \frac{\omega u_0}{\beta^2 c_0^2}(x-k_1\chi) - \frac{\omega w_0}{\beta^2 c_0^2}(r\sin\varphi - s\sin\delta_0) + \right.$$

$$\left. + \frac{\omega D_0}{\beta^2 c_0}\right) \cdot \left[D_0\left(1 - \frac{w_0 \omega s}{\beta^2 c_0^2}\cos\delta_0\right) + \frac{\omega r s}{\beta^2 c_0}\left(1 - \frac{u_0^2}{c_0^2}\right)\sin(\varphi-\delta_0) + \right.$$

$$\left. + \frac{\omega s}{\beta^2 c_0}\frac{w_0^2}{c_0^2}\sin\delta_0(r\cos\varphi - s\cos\delta_0) + \frac{\omega s}{\beta^2 c_0}\frac{u_0 w_0}{c_0^2}(x-k_1\chi)\cos\delta_0\right]^{-1};$$

with $$\delta_0 = \varphi_0 + \chi - \frac{\omega u_0}{\beta^2 c_0^2}(x-k_1\chi) - \frac{\omega w_0}{\beta^2 c_0^2}(r\sin\varphi - s\sin\delta_0) + \frac{\omega D_0}{\beta^2 c_0};$$

$$D_0 = \left[(x-k_1\chi)^2\left(1-\frac{w_0^2}{c_0^2}\right) + \left(1-\frac{u_0^2}{c_0^2}\right)(r^2+s^2-2rs\cos(\varphi-\delta_0)) - \right.$$

$$\left. - \frac{w_0^2}{c_0^2}(r\cos\varphi - s\cos\delta_0)^2 + 2\frac{u_0 w_0}{c_0^2}(x-k_1\chi)(r\sin\varphi - s\sin\delta_0)\right]^{1/2}. \quad (92)$$

As in normal propeller acoustics, the pressure field ρ radiated at the rotating airfoils by pressure discontinuity or buoyancy distribution, can be obtained on the blades through a dipole spread corresponding to the pressure drop. Then according to equations (84) and (85), ρ from a wave equation of the type (85) is sufficient. This model conforms to the linearized surface-support theory.

In the investigation presented here we have confined ourselves to the line-support theory (Nos. 2 to 6). The air force affected on the supporting line (per unit of length in the radial direction) is then to be described by a radial dipole distribution, whose dipole axis directions are perpendicular to the relative airflow velocity at the point of the supporting line. The unit vector in this dipole axis direction is obvious: /38

$$\mathfrak{N}_n = \frac{\left(s + \frac{w_0}{\omega}\cos\varphi_0\right)\mathfrak{N}_x - \frac{u_0}{\omega}\mathfrak{N}_\varphi}{\sqrt{\left(s+\frac{w_0}{\omega}\cos\varphi_0\right)^2 + \left(\frac{u_0}{\omega}\right)^2}}$$

With this we thereby obtain from the fundamental solution (92) through dipole formation with $\pi(S, \phi_0)$ as dipole moment the pressure field ρ radiated from the support distribution (or from the pressure drop) at all N rotor airfoils, in the form[11]:

[11] According to the line-support theory, x is to be equated to zero in equation (92).

$$p(x,r,\varphi,\varphi_0) = p(x,y,z,\varphi_0) = p_0 - \frac{1}{4\pi}\sum_{n=0}^{N-1}\int_{R_i}^{R_0}\left[\left(\dot{s}+\frac{w_0}{\omega}\cos\varphi_n\right)^2+\frac{u_0^2}{\omega^2}\right]^{-1/2}\cdot$$

$$\cdot\left[\left(s+\frac{w_0}{\omega}\cos\varphi_n\right)\frac{\partial}{\partial x}-\frac{u_0}{\omega s}\frac{\partial}{\partial\varphi}\right]\Pi(s,\sigma_n)\left[D_n\left(1-\frac{w_0\omega s}{\beta^2c_0^2}\cos\sigma_n\right)+\right.$$

$$+\frac{\omega r s}{\beta^2 c_0}\left(1-\frac{u_0^2}{c_0^2}\right)\sin(\varphi-\sigma_n)+\frac{\omega s}{\beta^2 c_0}\frac{w_0^2}{c_0^2}\sin\sigma_n(r\cos\varphi-s\cos\sigma_n)+$$

$$\left.+\frac{\omega s}{\beta^2 c_0}\frac{u_0 w_0}{c_0^2}x\cos\sigma_n\right]^{-1}ds; \qquad (93)$$

with
$$\sigma_n = \varphi_n - \frac{\omega u_0}{\beta^2 c_0^2}x - \frac{\omega w_0}{\beta^2 c_0^2}(r\sin\varphi - s\sin\sigma_n) + \frac{\omega D_n}{\beta^2 c_0};$$

$$D_n = \left[x^2\left(1-\frac{w_0^2}{c_0^2}\right)+\left(1-\frac{u_0^2}{c_0^2}\right)(r^2+s^2-2rs\cos(\varphi-\sigma_n))-\right.$$

$$\left.-\frac{w_0^2}{c_0^2}(r\cos\varphi-s\cos\sigma_n)^2+2\frac{u_0 w_0}{c_0^2}x(r\sin\varphi-s\sin\sigma_n)\right]^{1/2};$$

$$\beta^2 = 1-\frac{u_0^2}{c_0^2}-\frac{w_0^2}{c_0^2}.$$

For the formation of the derivatives of x and ϕ there results, through differentiation of D_n and σ_n with respect to x, ϕ and elimination, the relationship:

$$\frac{\partial D_n}{\partial x} = \left[x\left(1-\frac{w_0^2}{c_0^2}-\frac{w_0\omega s}{c_0^2}\cos\sigma_n\right)+\frac{rs\omega u_0}{c_0^2}\sin(\varphi-\sigma_n)+\right.$$

$$\left.+\frac{u_0 w_0}{c_0^2}(r\sin\varphi-s\sin\sigma_n)\right]\frac{1}{Ve}; \qquad (94)$$

$$\frac{\partial\sigma_n}{\partial x} = \frac{\omega}{\beta^2 c_0}\left[x\left(1-\frac{w_0^2}{c_0^2}\right)+\frac{u_0 w_0}{c_0^2}(r\sin\varphi-s\sin\sigma_n)-\frac{u_0}{c_0}D_n\right]\frac{1}{Ve}$$

$$\frac{\partial D_n}{\partial\varphi} = \left[x\frac{u_0 w_0}{c_0^2}r\cos\varphi+\frac{w_0^2}{c_0^2}(r\cos\varphi-s\cos\sigma_n)r\sin\varphi+\right.$$

$$\left.+rs\left(1-\frac{u_0^2}{c_0^2}\right)\sin(\varphi-\sigma_n)+\frac{rs\omega w_0}{c_0^2}\sin(\varphi-\sigma_n)(r\cos\varphi-s\cos\sigma_n)\right]\frac{1}{Ve}; \qquad (95)$$

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$$\frac{\partial\sigma_n}{\partial\varphi} = \frac{\omega}{\beta^2 c_0}\left[x\frac{u_0 w_0}{c_0^2}r\cos\varphi+\frac{w_0^2}{c_0^2}(r\cos\varphi-s\cos\sigma_n)r\sin\varphi+\right.$$

$$\left.+rs\left(1-\frac{u_0^2}{c_0^2}\right)\sin(\varphi-\sigma_n)-\frac{w_0}{c_0}D_n r\cos\varphi\right]\frac{1}{Ve};$$

$$Ne = D_m\left(1-\frac{w_0\omega s}{\beta^2 c_0^2}\cos\delta_m\right)+\frac{\omega r s}{\beta^2 c_0}\left(1-\frac{u_0^2}{c_0^2}\right)\sin(\varphi-\delta_m)+ \tag{96}$$
$$+\frac{\omega s}{\beta^2 c_0}\frac{w_0^2}{c_0^2}\sin\delta_m(r\cos\varphi-s\cos\delta_m)+\frac{\omega s}{\beta^2 c_0}\frac{u_0 w_0}{c_0^2}x\cos\delta_m .$$

In utilizing the integrands from formula (93) for space points of the far field (in which the radiated acoustic pressure field is of interest), no difficulties are encountered, since all the integrands are constant.

In order to obtain the velocity potential from equation (93) we have to integrate the relationship (84). First, equation (84) changes with the transformation (86) into:

$$\frac{1}{\rho_0}\left[p(\bar{x}\cos\lambda - Z\sin\lambda, y, \bar{x}\sin\lambda + Z\cos\lambda, \varphi_0) - p_0\right] = \omega\frac{\partial\Phi}{\partial\varphi_0} - U\frac{\partial\Phi}{\partial\bar{x}} ,$$

with the familiar solution

$$\Phi(\bar{x}, y, Z, \varphi_0) = \Phi(x, y, z, \varphi_0) = -\frac{1}{U\rho_0}\int_{-\infty}^{\bar{x}}\left[p(\bar{x}'\cos\lambda - Z\sin\lambda, y,\right.$$
$$\left.\bar{x}'\sin\lambda + Z\cos\lambda, \varphi_0 + \frac{\omega}{U}(\bar{x}-\bar{x}')) - p_0\right]d\bar{x}' =$$
$$= -\frac{1}{U\rho_0}\int_{-\infty}^{x\cos\lambda+z\sin\lambda}\left[p(\bar{x}'\cos\lambda + x\sin^2\lambda - z\sin\lambda\cos\lambda, y, \bar{x}'\sin\lambda -\right.$$
$$\left.- x\sin\lambda\cos\lambda + z\cos^2\lambda, \varphi_0 + \frac{\omega}{U}(x\cos\lambda + z\sin\lambda - \bar{x}')) - p_0\right]d\bar{x}' .$$

We apply

$$x\cos\lambda + z\sin\lambda - \bar{x}' = \frac{U}{\omega}\psi ,$$

thus the clear result follows:

$$\phi(x, y, z, \varphi_0) = -\frac{1}{\omega\rho_0}\int_0^{\infty}\left[p(x - \frac{u_0}{\omega}\psi, y, z - \frac{w_0}{\omega}\psi, \varphi_0+\psi) - p_0\right]d\psi \tag{97}$$

for the connection between the pressure field and the velocity potential.

If we now examine once more the case of incompressible flow, it follows from (93) and (97), with $c_o \to \infty$: /40

$$\Phi = -\frac{1}{4\pi}\frac{1}{\omega\varrho}\sum_{n=0}^{N-1}\int_{R_i}^{R_o}\int_{0}^{\infty}\frac{\Pi(s,\varphi_n+\psi)}{\sqrt{\left(s+\frac{w_o}{\omega}\cos(\varphi_n+\psi)\right)^2+\left(\frac{u_o}{\omega}\right)^2}}\cdot$$

$$\cdot\left[\left(x-\frac{u_o}{\omega}\psi\right)^2+\left(y-s\cos(\varphi_n+\psi)\right)^2+\left(z-s\sin(\varphi_n+\psi)-\frac{w_o}{\omega}\psi\right)^2\right]^{-3/2}\cdot \tag{98}$$

$$\cdot\left[\left(s+\frac{w_o}{\omega}\cos(\varphi_n+\psi)\right)\left(x-\frac{u_o}{\omega}\psi\right)+\frac{u_o}{\omega}y\sin(\varphi_n+\psi)-\right.$$

$$\left.-\frac{u_o}{\omega}\left(z-\frac{w_o}{\omega}\psi\right)\cos(\varphi_n+\psi)\right]d\psi\,ds\,.$$

Formula (98) corresponds exactly to the known (See [1]) velocity potential of the linearized line-support theory of the helicopter with $k_o = u_o/\omega$ and $k_* = W_o/\omega$ as well as for the airfoil circulation Γ of the relationship:

$$\Pi(s,\varphi_o) = \varrho\sqrt{(\omega s + w_o\cos\varphi_o)^2 + u_o^2}\cdot\Gamma(s,\varphi_o)\,. \tag{99}$$

Equation (99) describes exactly the Kutta-Joukowski principle. Then, as said above, the force per unit length in a radial direction conforms to the dipole moment Π at the supporting line.

Since from the solution of the integral equation for the boundary condition according to No. 5 the airfoil circulation Γ can be regarded as known, there also results from equation (99) the dipole moment needed for calculating the acoustic far field.

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